

# GENERATORS OF DETAILED BALANCE QUANTUM MARKOV SEMIGROUPS

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## Abstract

For a quantum Markov semigroup  $\mathcal{T}$  on the algebra  $\mathcal{B}(\mathfrak{h})$  with a faithful invariant state  $\rho$ , we can define an adjoint  $\tilde{\mathcal{T}}$  with respect to the scalar product determined by  $\rho$ . In this paper, we solve the open problems of characterising adjoints  $\tilde{\mathcal{T}}$  that are also a quantum Markov semigroup and satisfy the detailed balance condition in terms of the operators  $H, L_k$  in the Gorini Kossakowski Sudarshan Lindblad representation  $\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_k (L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k)$  of the generator of  $\mathcal{T}$ . We study the adjoint semigroup with respect to both scalar products  $\langle a, b \rangle = \text{tr}(\rho a^* b)$  and  $\langle a, b \rangle = \text{tr}(\rho^{1/2} a^* \rho^{1/2} b)$ .

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## 1 Introduction

The principle of detailed balance is at the basis of equilibrium physics. The notion of detailed balance for open quantum systems (Alicki[2], Frigerio and Gorini[8], Kossakowski, Frigerio, Gorini and Verri[10], Alicki and Lendi [3]) when the evolution is described by a uniformly continuous Quantum Markov Semigroup (QMS)  $\mathcal{T}$  with a faithful normal invariant state  $\rho$ , is formulated as a property of the generator  $\mathcal{L}$ .

Indeed, for a system with associated separable Hilbert space  $\mathfrak{h}$ , this can be written in the Gorini Kossakowski Sudarshan Lindblad (GKSL) form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_k (L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k) \quad (1)$$

where  $H$  and  $L_k$  are operators on  $\mathfrak{h}$  that can always be chosen satisfying  $\text{tr}(\rho L_k) = 0$  and the natural summability and minimality conditions (see Theorem 9). The state  $\rho$  defines a scalar product  $\langle x, y \rangle = \text{tr}(\rho x^* y)$  on the algebra  $\mathcal{B}(\mathfrak{h})$  of operators on  $\mathfrak{h}$ , and  $\mathcal{T}$  admits a dual semigroup with respect to this scalar product if there exists another uniformly continuous QMS  $\tilde{\mathcal{T}}$  (generated by  $\tilde{\mathcal{L}}$ ) such that  $\text{tr}(\rho \tilde{\mathcal{T}}_t(x)y) = \text{tr}(\rho x \mathcal{T}_t(y))$ . The QMS  $\mathcal{T}$  satisfies the quantum detailed balance condition if the effective Hamiltonian  $H$  commutes with  $\rho$  and  $\tilde{\mathcal{L}} = \mathcal{L} - 2i[H, \cdot]$  i.e. the dissipative part  $\mathcal{L}_0 = \mathcal{L} - i[H, \cdot]$  of  $\mathcal{L}$  is self-adjoint.

This generalizes the notion of detailed balance (reversibility) for a classical Markov semigroup which is called reversible when it is self-adjoint in the  $L^2$  space of an invariant measure. It is worth noticing, however, that, in the commutative case, the adjoint (dual) of a Markov semigroup is always a Markov semigroup.

The dual of a QMS  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  with respect to the state  $\rho$  may not be a QMS because the adjoint  $\tilde{\mathcal{T}}_t$  of the map  $\mathcal{T}_t$  may not be positive or, even more, may not be a  $*$ -map i.e.  $\tilde{\mathcal{T}}_t(a)^* \neq \tilde{\mathcal{T}}_t(a^*)$ . It is known (see e.g. Ref.[10] Prop. 2.1) that  $\tilde{\mathcal{T}}_t$  is a completely positive map if it commutes with the modular group  $(\sigma_t)_{t \in \mathbb{R}}$  ( $\sigma_t(a) = \rho^{it} a \rho^{-it}$ ) associated with  $\rho$ . More recently, Majewski and Streater[11] (Thm.6 p.7985) showed that the  $\tilde{\mathcal{T}}_t$  are (completely) positive whenever they are  $*$ -maps.

The structure of the generator (1) of a detailed balance QMS was studied in Ref.[2] and Ref.[10] under the additional assumption that it is a normal operator, i.e.  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  commute.

In this paper, we solve the open problems of characterising in terms of the operators  $H, L_k$  in (1) dual semigroups  $\tilde{\mathcal{T}}$  that are QMSs and when they satisfy the detailed balance condition without additional assumptions on the generator  $\mathcal{L}$ . The main results of this paper, Theorems 26 and 30, describe the structure of generators of QMS whose dual is still a QMS and, among them, the structure of those satisfying the detailed balance condition.

The dual semigroup is a QMS if and only if the maps  $\mathcal{T}_t$  commute with the modular automorphism (Theorem 8). When this happens we can find particular GKSL representations of  $\mathcal{L}$  as in (1), that we call privileged, with  $H$  commuting with  $\rho$  and the  $L_k, L_k^*$ 's eigenvalues of the modular automorphism (Definition 20) i.e.  $\rho L_k \rho^{-1} = \lambda_k L_k$ . Moreover, the generator  $\tilde{\mathcal{L}}$  of the dual semigroup admits a privileged GKSL representation with  $\tilde{H} = -H - c$  ( $c$  is a real constant) and  $\tilde{L}_k = \lambda_k^{-1/2} L_k^*$  (Theorem 26).

Finally the quantum detailed balance condition  $\mathcal{L} - \tilde{\mathcal{L}} = 2i[K, \cdot]$  (for some self-adjoint operator  $K$ ) holds if and only if  $H = K + c$  and there exists a unitary matrix  $(u_{k\ell})_{k\ell}$  such that  $\lambda_k^{-1/2} L_k^* = \sum_{\ell} u_{k\ell} L_{\ell}$  (Theorem 30).

There are other choices of the scalar product on  $\mathcal{B}(\mathfrak{h})$  induced by  $\rho$ ; we can define  $\langle a, b \rangle_s = \text{tr}(\rho^{1-s} a^* \rho^s b)$  for any  $s \in [0, 1]$ . The most studied case is the previous one with  $s = 0$ . The case  $s = 1/2$ , sometimes called *symmetric*, however, is also interesting (see Goldstein and Lindsay[9]). Indeed, as wrote Accardi and Mohari[1] (p.409), “it is worth characterizing the class of Markov semigroup such that  $\mathcal{T}_t = \tilde{\mathcal{T}}_t$ ” in full generality also for the dual semigroup with respect to the “symmetric” scalar product  $\langle a, b \rangle = \text{tr}(\rho^{1/2} a^* \rho^{1/2} b)$  (Petz’s duality). Note that the “symmetric” dual semigroup is always a QMS.

In Section 7 we solve this problem and the more general problem of characterising QMSs satisfying the “symmetric” detailed balance condition  $\mathcal{L} - \mathcal{L}' = 2i[K, \cdot]$  ( $\mathcal{L}'$  is the generator of the symmetric dual QMS). We show that the “symmetric” detailed balance is weaker than usual detailed balance (Proposition 39) and establish the relationships among the  $L_k$ ’s, the operator  $G = -2^{-1} \sum_k L_k^* L_k - iH$  and  $\rho$  of symmetric detailed balance  $\mathcal{L}$  (Theorem 40). Examples 38 and 41 show that, in the “symmetric” case, the effective Hamiltonian  $H$  may not commute with  $\rho$ .

The paper is organised as follows. In Section 2 we outline the detailed balance condition for classical Markov semigroups. Then we explore several possible definitions of the dual semigroup in Section 3 and study the generators of QMS whose dual is still a QMS in Section 4. In Section 5 we characterise generators of quantum detailed balance QMS. The special case of QMS on  $2 \times 2$  matrices is analysed in Section 6; in this case it turns out that, if the dual semigroup is a QMS, then it satisfies the quantum detailed balance condition. Further examples of detailed balance QMSs, also with unbounded generators can be found in the literature and in Ref.[7]. Finally, in Section 7, we study the symmetric detailed balance condition.

## 2 Classical detailed balance

Let  $(E, \mathcal{E}, \mu)$  be a measure space with  $\mu$   $\sigma$ -finite and let  $T = (T_t)_{t \geq 0}$  be a weakly\* continuous Markov semigroup of bounded positive linear maps on  $L^\infty(E, \mathcal{E}, \mu)$ .  $T$  is the dual semigroup of a strongly continuous contraction semigroup on the predual space  $L^1(E, \mathcal{E}, \mu)$  denoted  $T_*$ . Suppose that  $T$  admits a  $T$ -invariant probability density  $\pi$  (a norm one, non-negative, function in  $L^1(E, \mathcal{E}, \mu)$  such that  $T_{*t}\pi = 0$  for  $t \geq 0$ ) vanishing only on an element of  $\mathcal{E}$  of measure 0. Then, it is well-known that the sesquilinear form

$$(f, g) = \int_E \bar{f} g \pi \, d\mu$$

defines a scalar product on  $L^\infty(E, \mathcal{E}, \mu)$ , that we denote by  $\langle \cdot, \cdot \rangle_\pi$ , and putting

$$\tilde{T}_t(g) = \pi^{-1} T_{*t}(\pi g) \tag{2}$$

for each  $t \geq 0$  one defines the adjoint of the operator  $T_t$  with respect to this scalar product. Indeed,  $\pi g$  belongs to  $L^1(E, \mathcal{E}, \mu)$  and the  $T_*$ -invariance of  $\pi$  yields

$$|\tilde{T}_t(g)| \leq \pi^{-1} T_{*t}(\pi) \|g\|_\infty = \|g\|_\infty,$$

so that  $\tilde{T}_t$  is a well defined bounded operator on  $L^\infty(E, \mathcal{E}, \mu)$ . Moreover, we have

$$\begin{aligned}\langle \tilde{T}_t(g), f \rangle_\pi &= \int_E \overline{\tilde{T}_t(g)} f \pi d\mu = \int_E \pi^{-1} \overline{T_{*t}(\pi g)} f \pi d\mu \\ &= \int_E T_{*t}(\pi \bar{g}) f d\mu = \int_E (\pi \bar{g}) T_t(f) d\mu(x) = \langle g, T_t(f) \rangle_\pi\end{aligned}$$

for every  $f, g \in L^\infty(E, \mu)$ . Clearly the maps  $\tilde{T}_t$  are also positive, thus  $\tilde{T} = (\tilde{T}_t)_{t \geq 0}$  is a weakly\* continuous semigroup of bounded positive maps on  $L^\infty(E, \mathcal{E}, \mu)$ .

Finally, the semigroup  $\tilde{T}$  is Markov since  $\tilde{T}_t(\mathbf{1}) = \pi^{-1} T_{*t}(\pi) = \mathbf{1}$ .

**Definition 1** *We say that  $T$  satisfies classical detailed balance if every operator  $T_t$  is selfadjoint with respect to  $\langle \cdot, \cdot \rangle_\pi$ , i.e.  $\tilde{T}_t = T_t$ .*

Therefore,  $T$  satisfies classical detailed balance if and only if

$$T_t(f) = \pi^{-1} T_{*t}(\pi f). \quad (3)$$

**Remark 2** Detailed balance is equivalent to reversibility of classical Markov chains. Indeed, when  $E = \{1, \dots, d\}$  is a finite (for simplicity) set, endowed with the discrete  $\sigma$ -algebra  $\mathcal{E}$  and the counting measure  $\mu$ , with a Markov semigroup  $(T_t)_{t \geq 0}$  we can associate the transition rate matrix  $(q_{jk})_{1 \leq j, k \leq d}$  defined by

$$q_{jk} = \lim_{t \rightarrow 0} t^{-1} (T_t 1_{\{k\}} - 1_{\{k\}})(j)$$

( $1_{\{k\}}$  denotes the indicator function of the set  $\{k\}$ ). Denoting  $(\tilde{q}_{jk})_{1 \leq j, k \leq d}$  the transition rate matrix associated with the Markov semigroup  $(\tilde{T}_t)_{t \geq 0}$ , it follows immediately from the definitions that (3) is equivalent to the classical condition  $\pi_j q_{jk} = \pi_k q_{kj}$  for all  $j, k \in E$  called *reversibility*.

The same condition also arises in discrete time Markov chains.

### 3 The quantum dual semigroup

The definition of detail balance involves the dual semigroup with respect to the scalar product determined by the invariant state. When studying the non-commutative analogue two fundamental differences with the classical commutative case arise: 1) there are several possible dualities, 2) the dual semigroup might not be positive. In this section we analyze these problems.

Let  $\mathfrak{h}$  be a complex separable Hilbert space and let  $\mathcal{T}$  be a uniformly continuous QMS on  $\mathcal{B}(\mathfrak{h})$  generated by a bounded linear operator  $\mathcal{L}$ . A faithful invariant state  $\rho$  for  $\mathcal{T}$  can be written in the form

$$\rho = \sum_{k \geq 1} \rho_k |e_k\rangle \langle e_k|, \quad (4)$$

where  $\rho_k > 0$  for every  $k$ ,  $\sum_{k \geq 1} \rho_k = 1$  and  $(e_k)_{k \geq 1}$  is an orthonormal basis of  $\mathfrak{h}$ . Therefore  $\rho$  is invertible but, when  $\dim \mathfrak{h} = \aleph_0$ , its inverse  $\rho^{-1} = \sum_{k \geq 1} \rho_k^{-1} |e_k\rangle \langle e_k|$  is a positive operator with dense domain  $\rho(\mathfrak{h})$ .

**Definition 3** Let  $s \in [0, 1]$  fixed. We say that  $\mathcal{T}$  admits the  $s$ -dual semigroup with respect to  $\rho$  if there exists a uniformly continuous semigroup  $\tilde{\mathcal{T}} = \{\tilde{\mathcal{T}}_t\}_t$  on  $\mathcal{B}(\mathfrak{h})$  such that

$$\mathrm{tr}(\rho^{1-s}\tilde{\mathcal{T}}_t(a)\rho^s b) = \mathrm{tr}(\rho^{1-s}a\rho^s\mathcal{T}_t(b)) \quad (5)$$

for all  $a, b \in \mathcal{B}(\mathfrak{h})$ ,  $t \geq 0$ .

When  $s = 0$  we shall abbreviate the name of  $\tilde{\mathcal{T}}$  speaking of dual semigroup. We denote by  $\mathcal{T}_{*t}$  and  $\tilde{\mathcal{T}}_{*t}$  the predual maps of  $\mathcal{T}_t$  and  $\tilde{\mathcal{T}}_t$  respectively.

We remark that for every  $s \in [0, 1]$  the sesquilinear form

$$\langle a, b \rangle_s := \mathrm{tr}(\rho^{1-s}a^*\rho^s b)$$

defines a scalar product on  $\mathcal{B}(\mathfrak{h})$ : indeed

$$\langle a, a \rangle_s = \mathrm{tr}((\rho^{s/2}a\rho^{(1-s)/2})^*(\rho^{s/2}a\rho^{(1-s)/2})) \geq 0$$

and  $\langle a, a \rangle_s = 0$  implies  $\rho^{s/2}a\rho^{(1-s)/2} = 0$ , i.e.  $a = 0$  because  $\rho$  is invertible.

If  $\tilde{\mathcal{T}}_t$  is a  $*$ -map, then it is exactly the adjoint operator of  $\mathcal{T}_t$  with respect to the scalar product  $\langle \cdot, \cdot \rangle_s$ .

In our framework, we will always suppose that  $\mathcal{T}$  admits the  $s$ -dual semigroup.

**Proposition 4** For each  $t \geq 0$  and  $a \in \mathcal{B}(\mathfrak{h})$  we have

$$\rho^{1-s}\tilde{\mathcal{T}}_t(a)\rho^s = \mathcal{T}_{*t}(\rho^{1-s}a\rho^s). \quad (6)$$

Moreover, the following properties hold:

1.  $\tilde{\mathcal{T}}_t(\mathbf{1}) = \mathbf{1}$ ;
2.  $\tilde{\mathcal{T}}_{*t}(\rho) = \rho$ ;
3. if  $\tilde{\mathcal{T}}_t$  is positive, then it is also normal.

**Proof.** The identity (6) is easily checked starting from (5) and using that  $\mathrm{tr}(\rho^{1-s}a\rho^s\mathcal{T}_t(b)) = \mathrm{tr}(\mathcal{T}_{*t}(\rho^{1-s}a\rho^s)b)$ .

Putting  $a = \mathbf{1}$ , we find then  $\rho^{1-s}\tilde{\mathcal{T}}_t(\mathbf{1})\rho^s = \mathcal{T}_{*t}(\rho) = \rho$  by the invariance of  $\rho$ ; this implies  $(\tilde{\mathcal{T}}_t(\mathbf{1}) - \mathbf{1})\rho^s = 0$ , i.e.  $\tilde{\mathcal{T}}_t(\mathbf{1}) = \mathbf{1}$  for the density of  $\rho(\mathfrak{h})$  in  $\mathfrak{h}$ .

Taking  $b = \mathbf{1}$  in (5) yields  $\mathrm{tr}(\tilde{\mathcal{T}}_t(a)\rho) = \mathrm{tr}(a\rho)$  for all  $a \in \mathcal{B}(\mathfrak{h})$ . This means in particular that the map  $a \mapsto \mathrm{tr}(\tilde{\mathcal{T}}_t(a)\rho)$  is weakly\*-continuous, so  $\rho$  belongs to the domain of  $\tilde{\mathcal{T}}_{*t}$  and  $\tilde{\mathcal{T}}_{*t}(\rho) = \rho$ .

To prove property 3 it is enough to show that, for every increasing net  $(x_\alpha)_\alpha$  of positive elements in  $\mathcal{B}(\mathfrak{h})$  with  $\sup_\alpha x_\alpha = x \in \mathcal{B}(\mathfrak{h})$ , we have

$$\lim_\alpha \langle u, \tilde{\mathcal{T}}_t(x_\alpha)u \rangle = \langle u, \tilde{\mathcal{T}}_t(x)u \rangle$$

for each  $u$  in a dense subspace of  $\mathfrak{h}$ .

So, let  $u \in \rho(\mathfrak{h})$ ; then  $u = \rho^{1-s}v = \rho^s w$  for some  $v, w \in \mathfrak{h}$ . Therefore, equation 6 implies

$$\begin{aligned} \lim_{\alpha} \langle u, \tilde{\mathcal{T}}_t(x_{\alpha})u \rangle &= \lim_{\alpha} \langle v, \rho^{1-s} \tilde{\mathcal{T}}_t(x_{\alpha}) \rho^s w \rangle = \lim_{\alpha} \langle v, \mathcal{T}_{*t}(\rho^{1-s} x_{\alpha} \rho^s) w \rangle \\ &= \langle v, \mathcal{T}_{*t}(\rho^{1-s} x \rho^s) w \rangle = \langle v, \rho^{1-s} \tilde{\mathcal{T}}_t(x) \rho^s w \rangle = \langle u, \tilde{\mathcal{T}}_t(x) u \rangle, \end{aligned}$$

since  $\mathcal{T}_{*t}$  is normal. (q.e.d.)

It is clear from (6) that

$$\tilde{\mathcal{T}}_t(a) = \rho^{-(1-s)} \mathcal{T}_{*t}(\rho^{1-s} a \rho^s) \rho^{-s} \quad (7)$$

on the dense subset  $\rho^s(\mathfrak{h}) = \rho(\mathfrak{h})$  of  $\mathfrak{h}$ , so that the  $1/2$ -dual semigroup is completely positive and then it is a QMS thanks to Proposition 4.3. However, for  $s \neq 1/2$ , contrary to what happens in the commutative case, the maps  $\tilde{\mathcal{T}}_t$  might not be positive. In this case  $\tilde{\mathcal{T}}$  is not a QMS (see Example 25).

**Remark 5** If  $\mathfrak{h}$  is finite-dimensional, then any uniformly continuous QMS  $\mathcal{T}$  on  $\mathcal{B}(\mathfrak{h})$  admits the  $s$ -dual semigroup, since equation 7 defines a uniformly continuous semigroup of bounded operators on  $\mathcal{B}(\mathfrak{h})$  satisfying  $\text{tr}(\rho^{1-s} \tilde{\mathcal{T}}_t(a) \rho^s b) = \text{tr}(\rho^{1-s} a \rho^s \mathcal{T}_t(b))$ .

The relationships between the generators  $\mathcal{L}$ ,  $\tilde{\mathcal{L}}$ ,  $\mathcal{L}_*$  and  $\tilde{\mathcal{L}}_*$ , of  $\mathcal{T}$ ,  $\tilde{\mathcal{T}}$ ,  $\mathcal{T}_*$  and  $\tilde{\mathcal{T}}_*$  respectively are easily deduced.

**Proposition 6** *The semigroups  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  satisfy (5) if and only if, for all  $a, b \in \mathcal{B}(\mathfrak{h})$ , we have*

$$\text{tr}(\rho^{1-s} \tilde{\mathcal{L}}(a) \rho^s b) = \text{tr}(\rho^{1-s} a \rho^s \mathcal{L}(b)). \quad (8)$$

*In this case, the following identity holds*

$$\rho^{1-s} \tilde{\mathcal{L}}(a) \rho^s = \mathcal{L}_*(\rho^{1-s} a \rho^s). \quad (9)$$

*Moreover, if  $\tilde{\mathcal{T}}$  is a QMS, then*

$$\rho^s \mathcal{L}(a) \rho^{1-s} = \tilde{\mathcal{L}}_*(\rho^s a \rho^{1-s}) \quad (10)$$

**Proof.** The identity (8) clearly follows differentiating (5) at  $t = 0$ . Conversely, the identity (8), implies that, for all  $n \geq 0$  we have

$$\text{tr}(\rho^{1-s} \tilde{\mathcal{L}}^n(a) \rho^s b) = \text{tr}(\rho^{1-s} \tilde{\mathcal{L}}^{n-1}(a) \rho^s \mathcal{L}(b)) = \dots = \text{tr}(\rho^{1-s} a \rho^s \mathcal{L}^n(b)).$$

Multiplying by  $t^n/n!$  and summing on  $n$ , we obtain (5) because  $\mathcal{T}_t = \sum_{n \geq 0} t^n \mathcal{L}^n/n!$  and  $\tilde{\mathcal{T}}_t = \sum_{n \geq 0} t^n \tilde{\mathcal{L}}^n/n!$ .

Finally (9) and (10) follow from (8) by the same arguments leading to the identity (6) starting from (5). (q.e.d.)

We now characterise QMSs with  $s$ -dual for  $s = 0$  which is still a QMS. To this end, we start recalling some basic ingredient of Tomita-Takesaki theory.

Let  $L^2(\mathfrak{h})$  be the space of Hilbert-Schmidt operators on  $\mathfrak{h}$ , with scalar product given by  $\langle x, y \rangle_{HS} = \text{tr}(x^*y)$ . If we set  $\Omega = \rho^{1/2} \in L^2(\mathfrak{h})$  and  $\pi_\rho(a) : L^2(\mathfrak{h}) \rightarrow L^2(\mathfrak{h})$  the left multiplication by  $a \in \mathcal{B}(\mathfrak{h})$ , then we obtain a representation of  $\mathcal{B}(\mathfrak{h})$  on  $L^2(\mathfrak{h})$  such that  $\Omega$  is a cyclic and separating vector, and  $\text{tr}(\rho a) = \langle \Omega, \pi_\rho(a)\Omega \rangle_{HS}$  for every  $a \in \mathcal{B}(\mathfrak{h})$ . Under these hypothesis, identifying  $\mathcal{B}(\mathfrak{h})$  with  $\pi_\rho(\mathcal{B}(\mathfrak{h}))$ , the modular operator  $\Delta$  (see section 2.5.2 of Ref.[6]) is defined on the dense set  $\mathcal{B}(\mathfrak{h})\rho^{1/2}$  by

$$\Delta a \rho^{1/2} = \rho a \rho^{-1} \rho^{1/2} = \rho a \rho^{-1/2},$$

whereas a calculation shows that the modular group  $(\sigma_t)_{t \in \mathbb{R}}$  on  $\mathcal{B}(\mathfrak{h})$  is given by

$$\sigma_t(a) = \rho^{it} a \rho^{-it}.$$

We recall that an element  $a$  in  $\mathcal{B}(\mathfrak{h})$  is analytic for  $(\sigma_t)_t$  if there exists a strip

$$I_\lambda = \{z \in \mathbb{C} \mid |\Im z| < \lambda\}$$

and a function  $f : I_\lambda \rightarrow \mathcal{B}(\mathfrak{h})$  such that:

1.  $f(t) = \sigma_t(a)$  for all  $t \in \mathbb{R}$ ;
2.  $I_\lambda \ni z \rightarrow \text{tr}(\eta f(z))$  is analytic for all  $\eta \in L^1(\mathfrak{h})$  or, equivalently,  $I_\lambda \ni z \rightarrow \langle u, f(z)v \rangle$  is analytic for all  $u, v \in \mathfrak{h}$ .

We denote by  $\mathcal{A}$  the set of all analytic elements for  $(\sigma_t)_t$ .

It is a well known fact (Proposition 5 of [4]) that  $\mathcal{A}\rho^{1/2}$  is a core for  $\Delta$  and  $\sigma_z(a) = \rho^{iz} a \rho^{-iz} \in \mathcal{B}(\mathfrak{h})$  for all  $a \in \mathcal{A}$  and  $z \in \mathbb{C}$ .

In particular, the modular automorphism  $\sigma_{-i}$  on  $\mathcal{B}(\mathfrak{h})$  is defined by  $\sigma_{-i}(a) = \rho a \rho^{-1}$  for all  $a \in \mathcal{A}$  and it satisfies the following property

**Lemma 7** *If  $\sigma_{-i}(a) = \alpha a$ , then we have  $\sigma_{-i}(a^*) = \alpha^{-1} a^*$  and  $\alpha = \text{tr}(\rho a a^*) / \text{tr}(\rho a^* a)$ . In particular, every eigenvalue of  $\sigma_{-i}$  is strictly positive.*

**Proof.** Let  $\sigma_{-i}(a) = \alpha a$ ; then  $\alpha \neq 0$  for  $\sigma_{-i}$  is invertible, and

$$\sigma_{-i}(a^*) = \rho a^* \rho^{-1} = (\sigma_{-i}^{-1}(a))^* = (\alpha^{-1} a)^* = \overline{\alpha^{-1}} a^*.$$

But  $\text{tr}(\rho a a^*) = \text{tr}(\rho a \rho^{-1} \rho a^*) = \alpha \text{tr}(\rho a^* a) = \alpha \text{tr}(\rho a^* a)$ , so that we obtain  $\alpha = \text{tr}(\rho a a^*) / \text{tr}(\rho a^* a)$  positive. Therefore,  $\sigma_{-i}(a^*) = \alpha^{-1} a^*$ . (q.e.d.)

We say that a linear bounded operator  $X$  on  $\mathcal{B}(\mathfrak{h})$  commute with  $\sigma_z$  for some  $z \in \mathbb{C}$  if  $X(\sigma_z(a)) = \sigma_z(X(a))$  for all  $a \in \mathcal{A}$ .

We can now show the following characterisation of QMSs whose 0-dual is still a semigroup of *positive* linear maps, i.e. a QMS, adapting an argument from Majewski and Streater[11] (proof of Theorem 6).

**Theorem 8** *The following conditions are equivalent:*

1.  $\tilde{\mathcal{T}}$  is a QMS;
2. any  $\mathcal{T}_t$  commutes with  $\sigma_{-i}$ ;
3.  $\mathcal{L}$  commutes with  $\sigma_{-i}$ .

*If the above conditions hold, also the maps  $\mathcal{T}_r$ ,  $\mathcal{T}_{*r}$ ,  $\tilde{\mathcal{T}}_r$ ,  $\tilde{\mathcal{T}}_{*r}$  and the generators  $\mathcal{L}$ ,  $\mathcal{L}_*$ ,  $\tilde{\mathcal{L}}$ ,  $\tilde{\mathcal{L}}_*$  commute with the homomorphisms  $\sigma_t$  for all  $t, r \geq 0$ .*

**Proof.** (1)  $\Rightarrow$  (3) If  $\tilde{\mathcal{T}}$  is a QMS, then, in particular,  $\tilde{\mathcal{T}}_r$  satisfies  $\tilde{\mathcal{T}}_r(a^*) = \tilde{\mathcal{T}}_r(a)^*$  for all  $a \in \mathcal{B}(\mathfrak{h})$ ; therefore, by (10) with  $s = 0$  and the same formula taking the adjoint we have

$$\mathcal{L}(a)\rho = \tilde{\mathcal{L}}_*(a\rho) \quad \rho\mathcal{L}(a) = \tilde{\mathcal{L}}_*(\rho a),$$

so that, replacing  $a$  by  $\rho a \rho^{-1}$ ,

$$\mathcal{L}(\rho a \rho^{-1}) = \tilde{\mathcal{L}}_*((\rho a \rho^{-1})\rho)\rho^{-1} = \rho\mathcal{L}_r(a)\rho^{-1}$$

for all  $a \in \mathcal{A}$ . This means  $\mathcal{L} \circ \sigma_{-i} = \sigma_{-i} \circ \mathcal{L}$  in the previous sense.

(3)  $\Rightarrow$  (2) By induction we can show that  $\mathcal{L}^n$  and  $\sigma_{-i}$  commute for every  $n \geq 0$ ; then, also  $\mathcal{T}_t$  commute with  $\sigma_{-i}$ , for  $\mathcal{T}_t = \exp(t\mathcal{L}) = \sum_{n \geq 0} t^n \mathcal{L}^n / n!$ .

(2)  $\Rightarrow$  (1) Let us define a contraction semigroup  $(\hat{\mathcal{T}}_r)_{r \geq 0}$  on  $L^2(\mathfrak{h})$  by

$$\hat{\mathcal{T}}_r(a\rho^{1/2}) = \mathcal{T}_r(a)\rho^{1/2}$$

for all  $a \in \mathcal{B}(\mathfrak{h})$  and  $r \geq 0$ . Indeed, since  $\mathcal{T}_r(a^*)\mathcal{T}_r(a) \leq \mathcal{T}_r(a^*a)$  by the 2-positivity of  $\mathcal{T}_r$ , we have

$$\|\hat{\mathcal{T}}_r(a\rho^{1/2})\|_{HS}^2 = \text{tr} \left( \rho^{1/2} \mathcal{T}_r(a^*) \mathcal{T}_r(a) \rho^{1/2} \right) \leq \text{tr} \left( \rho^{1/2} \mathcal{T}_r(a^*a) \rho^{1/2} \right) = \|a\rho^{1/2}\|_{HS}^2$$

by the invariance of  $\rho$  and the semigroup property follows from a straightforward algebraic computation. Condition (2) implies then

$$\begin{aligned} \hat{\mathcal{T}}_r(\Delta a \rho^{1/2}) &= \hat{\mathcal{T}}_r(\rho a \rho^{-1} \rho^{1/2}) = \mathcal{T}_r(\rho a \rho^{-1}) \rho^{1/2} = \rho \mathcal{T}_r(a) \rho^{-1} \rho^{1/2} \\ &= \Delta \mathcal{T}_r(a) \rho^{1/2} = \Delta \hat{\mathcal{T}}_r(a \rho^{1/2}) \end{aligned}$$

for all  $a \in \mathcal{A}$ , i.e. any map  $\hat{\mathcal{T}}_r$  commutes with  $\Delta$  ( $a\rho^{1/2}$  is a core for  $\Delta$ ). Therefore, by spectral calculus,  $\hat{\mathcal{T}}_r$  also commutes with  $\Delta^{it}$  for all  $t \in \mathbb{R}$ . It follows that  $\mathcal{T}_r$  commutes with  $\sigma_t$  for all  $t \geq 0$ . Thus

$$\begin{aligned} \text{tr}(\sigma_t(\mathcal{T}_{*r}(b))a) &= \text{tr}(\mathcal{T}_{*r}(b)\sigma_{-t}(a)) = \text{tr}(b\mathcal{T}_r(\sigma_{-t}(a))) \\ &= \text{tr}(b\sigma_{-t}(\mathcal{T}_r(a))) = \text{tr}(\sigma_t(b)\mathcal{T}_r(a)) \\ &= \text{tr}(\mathcal{T}_{*r}(\sigma_t(b))a) \end{aligned}$$

for all  $a, b$ , i.e. also  $\mathcal{T}_{*r}$  commutes with  $\sigma_t$ . Then, for all  $r \geq 0$  and  $t \in \mathbb{R}$ , we get

$$\rho^{it} \mathcal{T}_{*r}(b) \rho^{-it} = \sigma_t(\mathcal{T}_{*r}(b)) = \mathcal{T}_{*r}(\sigma_t(b)) = \mathcal{T}_{*r}(\rho^{it} b \rho^{-it}). \quad (11)$$



We want to show that this equation holds for  $b = \rho^{1/2}a\rho^{1/2}$  and for certain complex  $t$ . Since the maps

$$z \rightarrow \rho^{iz} = e^{iz \ln \rho}, \quad z \rightarrow \rho^{-iz} = e^{-iz \ln \rho}$$

are analytic on  $\Im z \leq 0$  and  $\Im z \geq 0$  respectively, and the operator

$$\rho^{i(t+is)} \rho^{1/2} a \rho^{1/2} \rho^{-i(t+is)} = \rho^{it} \rho^{-s+1/2} a \rho^{1/2+s} \rho^{-it}$$

is trace class into the strip  $1/2 \leq s \leq 1/2$ , both sides of equation (11) have an analytic continuation into this strip, so that  $\rho^{iz} \mathcal{T}_{*r}(b) \rho^{-iz} = \mathcal{T}_{*r}(\rho^{iz} b \rho^{-iz})$  holds for all complex  $z$  with  $|\Im z| \leq 1/2$  and  $b = \rho^{1/2} a \rho^{1/2}$ . Taking  $z = -i/2$ , we get

$$\rho^{1/2} \mathcal{T}_{*r}(\rho^{1/2} a \rho^{1/2}) \rho^{-1/2} = \mathcal{T}_{*r}(\rho^{1/2} \rho^{1/2} a \rho^{1/2} \rho^{-1/2}) = \mathcal{T}_{*r}(\rho a) = \rho \tilde{\mathcal{T}}_r(a). \quad (12)$$

Hence

$$\tilde{\mathcal{T}}_r(a) = \rho^{-1/2} \mathcal{T}_{*r}(\rho^{1/2} a \rho^{1/2}) \rho^{-1/2},$$

therefore any operator  $\tilde{\mathcal{T}}_r$  is completely positive and  $\tilde{\mathcal{T}}$  is a QMS.

The above arguments also prove the claimed commutation of semigroups, their generators and the homomorphisms  $\sigma_t$ . (q.e.d.)

## 4 The generator of the dual semigroup

Suppose now that the dual semigroup  $\tilde{\mathcal{T}}$  (for  $s = 0$ ) is a QMS with generator  $\tilde{\mathcal{L}}$ . In this section we find the relationship between the operators  $H, L_k$  and  $\tilde{G}, \tilde{L}_k$  which appear in the Lindblad representation of  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ .

To this end, we start recalling the following result from Parthasarathy[12] (Th. 30.16) on the representation of the generator of a uniformly continuous QMS in a special form of GKLS (Gorini, Kossakowski, Sudarshan, Lindblad) type.

**Theorem 9** *Let  $\mathcal{L}$  be the generator of a uniformly continuous QMS on  $\mathcal{B}(\mathfrak{h})$  and let  $\rho$  be any normal state on  $\mathcal{B}(\mathfrak{h})$ . Then there exist a bounded selfadjoint operator  $H$  and a sequence  $(L_k)_{k \geq 1}$  of elements in  $\mathcal{B}(\mathfrak{h})$  such that:*

1.  $\text{tr}(\rho L_k) = 0$  for each  $k \geq 1$ ,
2.  $\sum_{k \geq 1} L_k^* L_k$  is strongly convergent,
3. if  $\sum_{k \geq 0} |c_k|^2 < \infty$  and  $c_0 + \sum_{k \geq 1} c_k L_k = 0$  for scalars  $(c_k)_{k \geq 0}$  then  $c_k = 0$  for every  $k \geq 0$ ,
4.  $\mathcal{L}(a) = i[H, a] - \frac{1}{2} \sum_{k \geq 1} (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k)$  for all  $a \in \mathcal{B}(\mathfrak{h})$ .

Moreover, if  $H', (L'_k)_{k \geq 1}$  is another family of bounded operators in  $\mathcal{B}(\mathfrak{h})$  with  $H'$  selfadjoint, then it satisfies conditions (1)-(4) if and only if the lengths of sequences  $(L_k)_{k \geq 1}, (L'_k)_{k \geq 1}$  are equal and

$$H' = H + \alpha, \quad L'_k = \sum_j u_{kj} L_j$$

for some scalar  $\alpha$  and a unitary matrix  $(u_{kj})_{kj}$ .

In our framework  $\rho$  will always be a faithful normal  $\mathcal{T}$ -invariant state.

We now introduce a terminology in order to distinguish GKSL representations with properties (1) and (3) in Theorem 9 from standard GKSL representations.

**Definition 10** We call special GKSL representation with respect to a state  $\rho$  by means of the operators  $H, L_k$  any representation of  $\mathcal{L}$  satisfying conditions (1), ..., (4) of Theorem 9.

**Remark 11** Condition 3 of Theorem 9 means that  $\{\mathbf{1}, L_1, L_2, \dots\}$  is a set of linearly independent elements of  $\mathcal{B}(\mathfrak{h})$ . If  $\dim \mathfrak{h} = d$ , then the length of  $(L_k)_{k \geq 1}$  in a special GKSL representation of  $\mathcal{L}$  is at most  $d^2 - 1$ .

We recall that we can also write  $\mathcal{L}$  as

$$\mathcal{L}(a) = G^*a + aG + \sum_k L_k^* a L_k,$$

where  $G$  is the bounded operator on  $\mathfrak{h}$  defined by

$$G = -iH - \frac{1}{2} \sum_k L_k^* L_k. \quad (13)$$

**Remark 12** The last statement of Theorem 9 implies that, in a special GKSL representation of  $\mathcal{L}$ , the above operator  $G$  is unique up to a purely imaginary multiple of the identity operator. Indeed the operator  $G'$  defined as in (13) replacing  $H, L_k$  by  $H', L'_k$  satisfies

$$\begin{aligned} G' &= -iH - i\alpha - \frac{1}{2} \sum_{k,j,m} \bar{u}_{kj} u_{km} L_j^* L_m \\ &= -iH - i\alpha - \frac{1}{2} \sum_{j,m} \left( \sum_k \bar{u}_{kj} u_{km} \right) L_j^* L_m \\ &= -iH - i\alpha - \frac{1}{2} \sum_j L_j^* L_j = G - i\alpha \end{aligned}$$

because the matrix  $(u_{kj})_{kj}$  is unitary.

Let  $\mathfrak{k}$  be a Hilbert space with Hilbertian dimension equal to the length of the sequence  $(L_k)_k$  and let  $(f_k)$  be an orthonormal basis of  $\mathfrak{k}$ . Defining a linear bounded operator  $L : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{k}$  by  $Lu = \sum_k L_k u \otimes f_k$ , Theorem 9 takes the following form (Theorem 30.12 Ref.[12])

**Theorem 13** If  $\mathcal{L}$  is the generator of a uniformly continuous QMS on  $\mathcal{B}(\mathfrak{h})$ , then there exist an Hilbert space  $\mathfrak{k}$ , a bounded linear operator  $L : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{k}$  and a bounded selfadjoint operator  $H$  in  $\mathfrak{h}$  satisfying the following:

1.  $\mathcal{L}(x) = i[H, x] - \frac{1}{2}(L^*Lx - 2L^*(x \otimes \mathbb{1}_k)L + xL^*L)$  for all  $x \in \mathcal{B}(\mathfrak{h})$ ;
2. the set  $\{(x \otimes \mathbb{1}_k)Lu : x \in \mathcal{B}(\mathfrak{h}), u \in \mathfrak{h}\}$  is total in  $\mathfrak{h} \otimes \mathfrak{k}$ .

**Proof.** Letting  $Lu = \sum_k L_k u \otimes f_k$ , where  $(f_k)$  is an orthonormal basis of  $\mathfrak{k}$  and the  $L_k$  are as in Theorem 9, a simple calculation shows that condition (1) is fulfilled.

Suppose that there exists a non-zero vector  $\xi \in \{(x \otimes \mathbb{1}_k)Lu : x \in \mathcal{B}(\mathfrak{h}), u \in \mathfrak{h}\}^\perp$ ; then  $\xi = \sum_k v_k \otimes f_k$  with  $v_k \in \mathfrak{h}$  and

$$0 = \langle \xi, (x \otimes \mathbb{1}_k)Lu \rangle = \sum_k \langle v_k, xL_k u \rangle = \sum_k \langle L_k^* x^* v_k, u \rangle$$

for all  $x \in \mathcal{B}(\mathfrak{h})$ ,  $u \in \mathfrak{h}$ . Hence,  $\sum_k L_k^* x^* v_k = 0$ . Since  $\xi \neq 0$ , we can suppose  $\|v_1\| = 1$ ; then, putting  $p = |v_1\rangle\langle v_1|$  and  $x = py^*$ ,  $y \in \mathcal{B}(\mathfrak{h})$ , we get

$$0 = L_1^* y v_1 + \sum_{k \geq 2} \langle v_1, v_k \rangle L_k^* y v_1 = \left( L_1^* + \sum_{k \geq 2} \langle v_1, v_k \rangle L_k^* \right) y v_1. \quad (14)$$

Since  $y \in \mathcal{B}(\mathfrak{h})$  is arbitrary, equation (14) contradicts the linear independence of the  $L_k$ 's. Therefore the set in (2) must be total. (q.e.d.)

We now study the generator  $\mathcal{L}$  of QMS  $\mathcal{T}$  whose dual  $\tilde{\mathcal{T}}$  is a QMS. As a first step we find an explicit form for the operator  $G$  defined by (13).

**Proposition 14** *If  $\mathcal{L}(a) = G^*a + aG + \sum_j L_j^* a L_j$  is a special GKSL representation of  $\mathcal{L}$  and  $\rho$  is the  $\mathcal{T}$ -invariant state (4) then*

$$G^*u = \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|) e_k - \text{tr}(\rho G)u \quad (15)$$

$$Gv = \sum_{k \geq 1} \rho_k \mathcal{L}_*(|v\rangle\langle e_k|) e_k - \text{tr}(\rho G^*)v \quad (16)$$

for every  $u, v \in \mathfrak{h}$ .

**Proof.** Since  $\mathcal{L}(|u\rangle\langle v|) = |G^*u\rangle\langle v| + |u\rangle\langle Gv| + \sum_j |L_j^*u\rangle\langle L_j^*v|$ , letting  $v = e_k$  we have  $G^*u = |G^*u\rangle\langle e_k| e_k$  and

$$G^*u = \mathcal{L}(|u\rangle\langle e_k|) e_k - \sum_j \langle e_k, L_j e_k \rangle L_j^* u - \langle e_k, G e_k \rangle u.$$

Multiplying both sides by  $\rho_k$  and summing on  $k$ , we find then

$$\begin{aligned} G^*u &= \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|) e_k - \sum_{j,k} \rho_k \langle e_k, L_j e_k \rangle L_j^* u - \sum_{k \geq 1} \rho_k \langle e_k, G e_k \rangle u \\ &= \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|) e_k - \sum_j \text{tr}(\rho L_j) L_j^* u - \text{tr}(\rho G)u \end{aligned}$$

and (15) follows since  $\text{tr}(\rho L_j) = 0$ . Computing the adjoint of  $G$  we find immediately (16). (q.e.d.)

**Proposition 15** *Let  $\tilde{T}$  be the  $s$ -dual of a QMS  $\mathcal{T}$  with generator  $\tilde{\mathcal{L}}$ . If  $G$  and  $\tilde{G}$  are the operators (16) in some special GKSL representations of  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  then*

$$\tilde{G}\rho^s = \rho^s G^* + \left( \text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) \right) \rho^s. \quad (17)$$

Moreover, we have  $\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) = ic_\rho$  for some  $c_\rho \in \mathbb{R}$ .

**Proof.** The identities (16) and (9) yield

$$\begin{aligned} \tilde{G}\rho^s &= \sum_{k \geq 1} \tilde{\mathcal{L}}_*(\rho^s |v\rangle\langle \rho_k^{1-s} e_k|) \rho_k^s e_k - \text{tr}(\rho \tilde{G}^*) \rho^s v \\ &= \sum_{k \geq 1} \tilde{\mathcal{L}}_*(\rho^s (|v\rangle\langle e_k|) \rho^{1-s}) \rho^s e_k - \text{tr}(\rho \tilde{G}^*) \rho^s v \\ &= \sum_{k \geq 1} \rho^s \mathcal{L}(|v\rangle\langle e_k|) \rho^{1-s} \rho^s e_k - \text{tr}(\rho \tilde{G}^*) \rho^s v \\ &= \rho^s G^* v + \left( \text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) \right) \rho^s v. \end{aligned}$$

Therefore, we obtain (17).

Right multiplying equation (17) by  $\rho^{1-s}$  we have  $\tilde{G}\rho = \rho^s G^* \rho^{1-s} + \left( \text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) \right) \rho$ , so that taking the trace,

$$\begin{aligned} \text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) &= \text{tr}(\tilde{G}\rho) - \text{tr}(\rho^s G^* \rho^{1-s}) \\ &= \text{tr}(\tilde{G}\rho) - \text{tr}(G^* \rho) = -\overline{\left( \text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) \right)}; \end{aligned}$$

this proves that  $\text{tr}(\rho G) - \text{tr}(\rho \tilde{G}^*) = ic_\rho$  for some real constant  $c_\rho$ . (q.e.d.)

**Proposition 16** *Let  $\tilde{T}$  be the 0-dual of a QMS  $\mathcal{T}$  and let*

$$\mathcal{L}(a) = G^* a + \sum_j L_j^* a L_j + a G, \quad \tilde{\mathcal{L}}(a) = \tilde{G}^* a + \sum_j \tilde{L}_j^* a \tilde{L}_j + a \tilde{G}$$

*be special GKSL representations of  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ . Then:*

1.  $\tilde{G} = G^* + ic$  with  $c \in \mathbb{R}$ ,
2. both  $G$  and  $\tilde{G}$  commute with  $\rho$ ,
3.  $\sum_k L_k^* L_k$ ,  $\sum_k \tilde{L}_k^* \tilde{L}_k$ ,  $H$  and  $\tilde{H}$  commute with  $\rho$ .

**Proof.** (1) It follows by Proposition 15 for  $s = 0$  and Theorem 9, Remark 12. Indeed, in any special GKSL representations of  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ ,  $G$  and  $\tilde{G}$  are unique up to a purely imaginary multiple of the identity operator.

(2) Let  $G$  and  $\tilde{G}$  be the operators (16) and in the given special GKSL representations of  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$ . Since  $\tilde{\mathcal{L}}_*(\rho a) = \rho \mathcal{L}(a)$  holds (for  $\tilde{T}$  is a QMS), we have

$$\begin{aligned}\tilde{G}\rho v &= \sum_{k=1}^n \rho_k \tilde{\mathcal{L}}_*(\rho|v\rangle\langle \rho e_k|)e_k - \text{tr}(\rho \tilde{G}^*)\rho \\ &= \sum_{k=1}^n \rho_k \rho \mathcal{L}(|v\rangle\langle e_k|)e_k - \text{tr}(\rho \tilde{G}^*)\rho v \\ &= \rho(G^*v + \text{tr}(\rho G)v) - \text{tr}(\rho \tilde{G}^*)\rho v\end{aligned}$$

for every  $v \in \mathfrak{h}$ , that is  $\tilde{G}\rho = \rho G^* + \text{tr}(\rho G)\rho$ . But  $G$  and  $\tilde{G}$  are the operators (16), therefore by (1) we have also  $\tilde{G}\rho = G^*\rho + \text{tr}(\rho G)\rho$ , and so  $G^*\rho = \rho G^*$ . This, together with Remark 12, clearly implies (2).

(3) Follows from (2) by decomposing  $G$  and  $\tilde{G}$  into their self-adjoint and anti self-adjoint parts. (q.e.d.)

We now study the properties of the  $L_k$  when  $\tilde{T}$  is a QMS.

**Lemma 17** *With the notations of Theorem 13, if  $\mathfrak{h}$  is finite-dimensional then the equation*

$$\sum_k L_k^* a L_k = \sum_k \rho^{-1} L_k^* \rho a \rho^{-1} L_k \rho \quad \forall a \in \mathcal{B}(\mathfrak{h}) \quad (18)$$

*implies  $\rho L_k \rho^{-1} = \lambda_k L_k$  and  $\rho L_k^* \rho^{-1} = \lambda_k^{-1} L_k^*$  for some positive  $\lambda_k$ .*

**Proof.** Define two linear maps  $X_1, X_2$  on  $\mathfrak{h} \otimes \mathfrak{k}$  by

$$\begin{aligned}X_1(x \otimes \mathbb{1}_k)Lu &= (x \otimes \mathbb{1}_k)(\rho \otimes \mathbb{1}_k)L\rho^{-1}u \\ X_2(x \otimes \mathbb{1}_k)Lu &= (x \otimes \mathbb{1}_k)(\rho^{-1} \otimes \mathbb{1}_k)L\rho u\end{aligned}$$

for all  $x \in \mathcal{B}(\mathfrak{h})$  and  $u \in \mathfrak{h}$ .

We postpone to Lemma 18 the proof that  $X_1$  and  $X_2$  are well defined on the total (Theorem 13) set  $\{(x \otimes \mathbb{1}_k)Lu \mid x \in \mathcal{B}(\mathfrak{h}), u \in \mathfrak{h}\}$  in  $\mathfrak{h} \otimes \mathfrak{k}$ . We can now extend  $X_1$  and  $X_2$  to a bounded operator on  $\mathfrak{h} \otimes \mathfrak{k}$ . Moreover, (21) implies

$$\begin{aligned}\langle X_1(x \otimes \mathbb{1}_k)Lu, X_2(y \otimes \mathbb{1}_k)Lv \rangle &= \sum_k \langle u, \rho^{-1} L_k^* \rho x^* y \rho^{-1} L_k \rho v \rangle \\ &= \sum_k \langle u, L_k^* x^* y L_k v \rangle \\ &= \langle (x \otimes \mathbb{1}_k)Lu, (y \otimes \mathbb{1}_k)Lv \rangle\end{aligned}$$

for all  $x, y \in \mathcal{B}(\mathfrak{h})$  and  $u \in \mathfrak{h}$ . As a consequence we have  $X_1^* X_2 = \mathbb{1}_{\mathfrak{h} \otimes \mathfrak{k}}$ .

By the definition of  $X_1, X_2$  we have also  $X_j(y \otimes \mathbb{1}_k) = (y \otimes \mathbb{1}_k)X_j$  for  $j = 1, 2$ . Therefore  $X_j$  can be written in the form  $\mathbb{1}_{\mathfrak{h}} \otimes Y_j$  for some invertible bounded map  $Y_j$  on  $\mathfrak{k}$  satisfying  $Y_1^* Y_2 = \mathbb{1}_{\mathfrak{k}}$ .

The definition of  $X_1, X_2$  implies then

$$(\mathbb{1}_h \otimes Y_1)L = (\rho \otimes \mathbb{1}_k)L\rho^{-1}, \quad (\mathbb{1}_h \otimes Y_1^*)^{-1}L = (\rho^{-1} \otimes \mathbb{1}_k)L\rho, \quad (19)$$

right multiplying by  $\rho$  and left multiplying by  $(\rho \otimes \mathbb{1}_k)$  the first and the second identity we find

$$(\mathbb{1}_h \otimes Y_1)L\rho = (\rho \otimes \mathbb{1}_k)L, \quad (\mathbb{1}_h \otimes Y_1^*)^{-1}(\rho \otimes \mathbb{1}_k)L = L\rho.$$

Writing the second as  $(\rho \otimes \mathbb{1}_k)L = (\mathbb{1}_h \otimes Y_1^*)L\rho$  we obtain

$$(\mathbb{1}_h \otimes Y_1)L\rho = (\rho \otimes \mathbb{1}_k)L = (\mathbb{1}_h \otimes Y_1^*)L\rho.$$

Since  $\rho$  is faithful, it follows that  $(\mathbb{1}_h \otimes Y_1)L = (\mathbb{1}_h \otimes Y_1^*)L$  (and also  $(\mathbb{1}_h \otimes Y_1)(x \otimes \mathbb{1}_k)L = (\mathbb{1}_h \otimes Y_1^*)(x \otimes \mathbb{1}_k)L$  for all  $x$ ) proving that  $Y_1$  is self-adjoint.

Therefore, there exist non-zero  $\lambda_k \in \mathbb{R}$  and a unitary operator  $U$  on  $k$  such that  $Y_1 = U^*DU$  with  $D = \text{diag}(\lambda_1, \lambda_2, \dots)$ . The identities (19) yield then

$$\begin{aligned} (\mathbb{1}_h \otimes DU)L &= (\mathbb{1}_h \otimes U)(\rho \otimes \mathbb{1}_k)L\rho^{-1} = (\rho \otimes \mathbb{1}_k)(\mathbb{1}_h \otimes U)L\rho^{-1} \\ (\mathbb{1}_h \otimes D^{-1}U)L &= (\mathbb{1}_h \otimes U)(\rho^{-1} \otimes \mathbb{1}_k)L\rho = (\rho^{-1} \otimes \mathbb{1}_k)(\mathbb{1}_h \otimes U)L\rho \end{aligned}$$

Thus, putting  $L' = UL$ , or, more precisely  $L'_k = \sum_\ell u_{k\ell}L_\ell$  for all  $k$ , we have

$$\rho L'_k \rho^{-1} = \lambda_k L'_k \quad \text{and} \quad \rho^{-1} L'_k \rho = \lambda_k^{-1} L'_k$$

for every  $k$ . To conclude the proof it suffices to recall that  $\lambda_k > 0$  by Lemma 7, since the above identities mean that  $\lambda_k$  is an eigenvalue of  $\sigma_{-i}$ . (q.e.d.)

We now check that the maps  $X_1, X_2$  introduced in the proof of Lemma 17 are well defined.

**Lemma 18** *With the notations of Lemma 17, if  $h$  is finite-dimensional and equation (18) holds, then*

$$\sum_{j=1}^m (x_j \otimes \mathbb{1}_k) L u_j = 0 \quad (20)$$

for  $x_1, \dots, x_m \in \mathcal{B}(h)$ ,  $u_1, \dots, u_m \in h$  implies:

1.  $\sum_{j=1}^m (x_j \otimes \mathbb{1}_k)(\rho \otimes \mathbb{1}_k)L\rho^{-1}u_j = 0;$
2.  $\sum_{j=1}^m (x_j \otimes \mathbb{1}_k)(\rho^{-1} \otimes \mathbb{1}_k)L\rho u_j = 0.$

**Proof.** Suppose that (20) holds. Taking the adjoint of (21) we find

$$\sum_k \rho L_k^* \rho^{-1} a \rho L_k \rho^{-1} = \sum_k L_k^* a L_k$$

for every  $a \in \mathcal{B}(\mathfrak{h})$  and compute

$$\begin{aligned} & \langle (y\rho^{-1} \otimes \mathbb{1}_{\mathfrak{k}})L\rho v, \sum_{j=1}^m (x_j \otimes \mathbb{1}_{\mathfrak{k}})(\rho \otimes \mathbb{1}_{\mathfrak{k}})L\rho^{-1}u_j \rangle \\ &= \sum_{j=1}^m \sum_k \langle v, L_k^* y^* x_j L_k u_j \rangle = \langle (y \otimes \mathbb{1}_{\mathfrak{k}})Lv, \sum_{j=1}^m (x_j \otimes \mathbb{1}_{\mathfrak{k}})Lu_j \rangle = 0 \end{aligned}$$

for all  $y \in \mathcal{B}(\mathfrak{h})$  and  $v \in \mathfrak{h}$ . But the set  $S = \{(y\rho^{-1} \otimes \mathbb{1}_{\mathfrak{k}})L\rho v \mid y \in \mathcal{B}(\mathfrak{h}), v \in \mathfrak{h}\}$  is total in  $\mathfrak{h} \otimes \mathfrak{k}$ , because  $\{(y \otimes \mathbb{1}_{\mathfrak{k}})Lv \mid y \in \mathcal{B}(\mathfrak{h}), v \in \mathfrak{h}\}$  is total (Theorem 13) and the maps  $y \mapsto y\rho^{-1}$ ,  $v \mapsto \rho v$  are bijective. This proves (1). The proof of (2) is similar and we omit it. (q.e.d.)

**Proposition 19** *Suppose that  $\mathcal{L}$  and  $\sigma_{-i}$  commute. Then there exists a special GKSL representation of  $\mathcal{L}$  in which, for all  $k$ , we have*

$$\rho L_k = \lambda_k L_k \rho, \quad \rho L_k^* = \lambda_k^{-1} L_k^* \rho, \quad \lambda_k > 0.$$

**Proof.** Define  $p_n := \sum_{k \leq n} |e_k\rangle\langle e_k|$  for  $n \geq 0$  and consider  $u, v \in \mathfrak{h}$ ,  $a \in \mathcal{B}(\mathfrak{h})$ ; since the map

$$z \mapsto \langle u, \rho^{iz} p_n a p_n \rho^{-iz} \rangle = \sum_{k, h \leq n} \rho_k^{iz} \rho_h^{-iz} \langle u, e_k \rangle \langle e_k, a e_h \rangle \langle e_h, v \rangle$$

is analytic on  $\mathbb{C}$ , then  $b := p_n a p_n$  belongs to  $\mathcal{A}$  for all  $n \geq 0$ . As a consequence, since  $\mathcal{L}$  and  $\sigma_{-i}$  commute, we have  $\mathcal{L}(b) = \rho^{-1} \mathcal{L}(\rho b \rho^{-1}) \rho$ , so that

$$\begin{aligned} i[H, b] - \frac{1}{2} \sum_k (L_k^* L_k b - 2L_k^* b L_k + b L_k^* L_k) &= \rho^{-1} [H, \rho b \rho^{-1}] \rho \\ &\quad - \frac{1}{2} \sum_k (\rho^{-1} L_k^* L_k \rho b - 2\rho^{-1} L_k^* \rho b \rho^{-1} L_k \rho + b \rho^{-1} L_k^* L_k \rho). \end{aligned}$$

Both  $H$  and  $\sum_k L_k^* L_k$  commute with  $\rho$  by Proposition 16 (3). We have then

$$\sum_k L_k^* p_n a p_n L_k = \sum_k \rho^{-1} L_k^* \rho p_n a p_n \rho^{-1} L_k \rho, \quad (21)$$

and so

$$\sum_k p_n L_k^* p_n a p_n L_k p_n = \sum_k p_n \rho^{-1} L_k^* \rho p_n a p_n \rho^{-1} L_k \rho p_n$$

for all  $a \in \mathcal{B}(\mathfrak{h})$ ,  $n \geq 0$ , right and left multiplying (21) by  $p_n$ . Remembering that  $p_n \rho^{-1} = \rho^{-1} p_n$  on  $\rho(\mathfrak{h})$  and setting  $L_{(n)k} := p_n L_k p_n$ ,  $\rho_{(n)} := \rho p_n$ , the above equality gives

$$\sum_k \rho_{(n)}^{-1} L_{(n)k}^* \rho_{(n)} b \rho_{(n)}^{-1} L_{(n)k} \rho_{(n)} = \sum_k L_{(n)k}^* b L_{(n)k}$$

for all  $b \in \mathcal{B}(p_n(\mathfrak{h}))$ .

But  $\rho_{(n)}$  is faithful on the finite-dimensional Hilbert space  $p_n(\mathfrak{h})$  and  $\{(x \otimes \mathbb{1}_k)L_{(n)}u \mid x \in \mathcal{B}(p_n(\mathfrak{h})), u \in p_n(\mathfrak{h})\}$  is total in  $p_n(\mathfrak{h}) \otimes \mathfrak{k}$ , therefore Lemma 17 assures that

$$\rho_{(n)}L_{(n)k}\rho_{(n)}^{-1} = \lambda_{k,n}L_{(n)k} \quad \rho_{(n)}L_{(n)k}^*\rho_{(n)}^{-1} = \lambda_{k,n}^{-1}L_{(n)k}^*$$

for some  $\lambda_{k,n} > 0$ , i.e.

$$\rho p_n L_k p_n = \lambda_{k,n} p_n L_k \rho p_n, \quad \rho p_n L_k^* p_n = \lambda_{k,n}^{-1} p_n L_k^* \rho p_n. \quad (22)$$

Since  $(p_n)_n$  is an increasing sequence of projections, this implies  $\lambda_{k,n} = \lambda_k$  for  $n \gg 0$ , and then, letting  $n \rightarrow \infty$  in equation (22), we obtain

$$\rho L_k = \lambda_k L_k \rho, \quad \rho L_k^* = \lambda_k^{-1} L_k^* \rho,$$

for  $(p_n)_n$  converges to  $\mathbb{1}$  in the strong operator topology. (q.e.d.)

**Definition 20** Let  $\mathcal{L}$  be the generator of a QMS and let  $\rho$  be a faithful normal state. A special GKSL representation of  $\mathcal{L}$  with respect to  $\rho$  by means of operators  $H, L_k$  is called privileged if their operators  $L_k$  satisfy  $\rho L_k = \lambda_k L_k \rho$  and  $\rho L_k^* = \lambda_k^{-1} L_k^* \rho$  for some  $\lambda_k > 0$  and  $H$  commutes with  $\rho$ .

**Remark 21** The operator  $\sum_k L_k^* L_k$  (the self-adjoint part of  $G$ ) in a privileged GKSL representation clearly commutes with  $\rho$ . Moreover, the constants  $\lambda_k$  are determined by the eigenvalues of  $\rho$ . Indeed, writing  $\rho$  as in (4), the identity  $\rho L_k = \lambda_k L_k \rho$  yields

$$\rho_j \langle e_j, L_k e_m \rangle = \langle e_j, \rho L_k e_m \rangle = \lambda_k \langle e_j, L_k \rho e_m \rangle = \lambda_k \rho_m \langle e_j, L_k e_m \rangle.$$

Therefore  $\lambda_k = \rho_j \rho_m^{-1}$  for all  $j, m$  such that  $\langle e_j, L_k e_m \rangle \neq 0$ . In particular, if we write  $\rho = e^{-H_S}$  for some bounded selfadjoint operator  $H_S = \sum_i \varepsilon_i |e_i\rangle\langle e_i|$  on  $\mathfrak{h}$ , we find  $\lambda_k = e^{\varepsilon_m - \varepsilon_j}$ .

**Proposition 22** Given two privileged GKSL of  $\mathcal{L}$  with respect to the same state  $\rho$  by means of operators  $H, L_k$  and  $H', L'_k$ , with  $D = \text{diag}(\lambda_1, \lambda_2, \dots)$  and  $D' = \text{diag}(\lambda'_1, \lambda'_2, \dots)$ , there exists a unitary operator  $V$  on  $\mathfrak{k}$  and  $\alpha \in \mathbb{R}$  such that

$$H' = H + \alpha, \quad L' = (\mathbb{1}_{\mathfrak{h}} \otimes V)L, \quad D' = V D V^*.$$

**Proof.** By Theorem 9 there exist  $\alpha \in \mathbb{R}$  and a unitary  $V$  on  $\mathfrak{k}$  such that  $H' = H + \alpha$  and  $L' = (\mathbb{1}_{\mathfrak{h}} \otimes V)L$ . Since both families  $H, L_k$  and  $H', L'_k$  give privileged GKSL representations with respect to the same state  $\rho$ , we have

$$(\rho \otimes \mathbb{1}_{\mathfrak{k}})L = (\mathbb{1}_{\mathfrak{h}} \otimes D)L\rho, \quad (\rho \otimes \mathbb{1}_{\mathfrak{k}})L' = (\mathbb{1}_{\mathfrak{h}} \otimes D')L'\rho.$$

Left multiplying the first identity by  $(\mathbb{1}_{\mathfrak{h}} \otimes V)$  and replacing  $L'$  by  $VL$  in the second we find

$$(\rho \otimes \mathbb{1}_{\mathfrak{k}})(\mathbb{1}_{\mathfrak{h}} \otimes V)L = (\mathbb{1}_{\mathfrak{h}} \otimes V D)L\rho, \quad (\rho \otimes \mathbb{1}_{\mathfrak{k}})(\mathbb{1}_{\mathfrak{h}} \otimes V)L = (\mathbb{1}_{\mathfrak{h}} \otimes D'V)L\rho$$

It follows that  $VD = D'V$ , i.e.  $D' = V D V^*$ . (q.e.d.)



**Remark 23** The identity  $D' = VDV^*$  means that  $V$  is a change of coordinates that transforms  $D$  into another diagonal matrix; in particular, if  $D = \text{diag}(\lambda_1, \lambda_2, \dots)$  and  $D' = \text{diag}(\lambda'_1, \lambda'_2, \dots)$ , we have

$$\lambda'_i \langle f_i, Vf_j \rangle = \lambda_j \langle f_i, Vf_j \rangle.$$

Since  $V$  is a unitary operator this implies that, when the  $\lambda_k$  are all different, for every  $i$  there exists a unique  $j$  such that  $\langle f_i, Vf_j \rangle \neq 0$  and for every  $j$  there exists a unique  $i$  such that  $\langle f_i, Vf_j \rangle \neq 0$ . Thus

$$Vf_j = e^{i\theta_{\sigma(j)}} f_{\sigma(j)} \quad \text{and} \quad L'_k = e^{i\theta_{\sigma(k)}} L_{\sigma(k)}$$

with  $\theta_{\sigma(j)} \in \mathbb{R}$  and  $\sigma$  a permutation. Therefore, when the  $\lambda_k$  are all different, privileged GKSL representations of  $\mathcal{L}$ , if exist, are unique up to a permutation of the operators  $L_k$ , a multiplication of each  $L_k$  by a phase  $e^{i\theta_k}$  and a constant  $\alpha$  in the Hamiltonian  $H$ .

If some  $\lambda_k$ 's are equal, then also unitary transformations of subspaces of  $\mathfrak{k}$  associated with the same  $\lambda_k$ 's are allowed.

The results of this section are summarized by the following

**Theorem 24** *The 0-dual semigroup  $\tilde{\mathcal{T}}$  of a QMS  $\mathcal{T}$  generated by  $\mathcal{L}$  with faithful normal invariant state  $\rho$  is a QMS if and only there exists a privileged GKSL representation of  $\mathcal{L}$  with respect to  $\rho$ .*

**Proof.** If  $\tilde{\mathcal{T}}$  is a QMS, then  $\mathcal{L}$  commutes with  $\sigma_{-i}$  by Theorem 8 and so there exists a special GKSL representation of  $\mathcal{L}$  by Propositions 19 and 16.

The converse is trivial.

(q.e.d.)

We now exhibit an example of semigroup whose dual is not a QMS.

**Example 25** Consider the semigroup  $\mathcal{T}$  on  $M_2(\mathbb{C})$  generated by

$$\mathcal{L}(a) = i \frac{\Omega}{2} [\sigma_1, a] - \frac{\mu^2}{2} (\sigma^+ \sigma^- a - 2\sigma^+ a \sigma^- + a \sigma^+ \sigma^-),$$

where  $\mu > 0$ ,  $\Omega \in \mathbb{R}$ ,  $\Omega \neq 0$  and  $\sigma_k$  are the Pauli matrices and  $\sigma^\pm = \sigma_1 \pm i\sigma_2$  are the raising and lowering operator.

A straightforward computation shows the state

$$\rho = \frac{1}{2} \left( \mathbb{1} + \frac{2\mu^2\Omega}{2\Omega^2 + \mu^4} \sigma_2 - \frac{\mu^4}{2\Omega^2 + \mu^4} \sigma_3 \right) = \frac{1}{2\Omega^2 + \mu^4} \begin{pmatrix} \Omega^2 & -i\mu^2\Omega \\ i\mu^2\Omega & \Omega^2 + \mu^4 \end{pmatrix}$$

is invariant and faithful. The generator  $\mathcal{L}$  can be written in a special GKSL form (with respect to the invariant state  $\rho$ ) with

$$L_1 = \mu\sigma^- - \frac{\mu}{2} \text{tr}(\rho\sigma^-) \mathbb{1} = \mu\sigma^- + i \frac{\mu^3\Omega}{2(2\Omega^2 + \mu^4)}, \quad H = \left( \frac{\Omega}{2} + \frac{\mu^4\Omega}{2\Omega^2 + \mu^4} \right) \sigma_1.$$

The dual semigroup  $\tilde{\mathcal{T}}$  of  $\mathcal{T}$  is not a QMS because  $H$  does not commute with  $\rho$ .

We now establish the relationship between the privileged GKSL representations of a generator  $\mathcal{L}$  and its 0-dual  $\tilde{\mathcal{L}}$ .

**Theorem 26** *If  $\tilde{\mathcal{T}}$  is a QMS, then, for every privileged GKSL representation of  $\mathcal{L}$ , by means of operators  $H, L_k$ , there exists a privileged GKSL representation of  $\tilde{\mathcal{L}}$ , by means of operators  $\tilde{H}, \tilde{L}_k$  such that:*

1.  $\tilde{H} = -H - \alpha$  for some  $\alpha \in \mathbb{R}$ ;
2.  $\tilde{L}_k = \lambda_k^{-1/2} L_k^*$  for some  $\lambda_k > 0$ .

**Proof.** Consider a privileged GKSL representation of  $\mathcal{L}$

$$\mathcal{L}(a) = i[H, a] - \frac{1}{2} \sum_{k \geq 1} (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k),$$

with  $H\rho = \rho H$  and  $\rho L_k \rho^{-1} = \lambda_k L_k$ ,  $\rho L_k^* \rho^{-1} = \lambda_k^{-1} L_k^*$  for some  $\lambda_k > 0$ .

Since  $\rho \tilde{\mathcal{L}}(a) = \mathcal{L}_*(\rho a)$ , we have

$$\begin{aligned} \rho \tilde{\mathcal{L}}(a) &= -i[H, \rho a] - \frac{1}{2} \sum_k (\rho a L_k^* L_k - 2L_k \rho a L_k^* + L_k^* L_k \rho a) \\ &= -i\rho[H, a] - \frac{1}{2} \sum_k (\rho a L_k^* L_k - 2\lambda_k^{-1} \rho L_k a L_k^* + L_k^* L_k \rho a). \end{aligned}$$

But  $\rho$  is  $\mathcal{T}$ -invariant and commutes with  $H$  thus  $\sum_k \rho L_k^* L_k = \sum_k L_k^* L_k \rho = \sum_k L_k \rho L_k^* = \sum_k \lambda_k^{-1} \rho L_k L_k^*$ . It follows that  $\sum_k L_k^* L_k = \sum_k \lambda_k^{-1} L_k L_k^*$  and

$$\rho \tilde{\mathcal{L}}(a) = \rho \left( -i[H, a] - \frac{1}{2} \sum_k (a \lambda_k^{-1} L_k L_k^* - 2\lambda_k^{-1} L_k a L_k^* + \lambda_k^{-1} L_k L_k^* a) \right).$$

Therefore, putting  $\tilde{H} = -H - \alpha$  ( $\alpha \in \mathbb{R}$ ) and  $\tilde{L}_k = \lambda_k^{-1/2} L_k^*$ , we find a GKSL representation of  $\tilde{\mathcal{L}}$ .

Since  $[\tilde{H}, \rho] = 0$ ,  $\text{tr}(\rho \tilde{L}_k) = 0$  for every  $k$  and  $\{\mathbf{1}, \tilde{L}_k \mid k \geq 1\}$  is clearly a set of linearly independent elements, we found a special GKSL representation of  $\tilde{\mathcal{L}}$  by means of the operators  $\tilde{H}, \tilde{L}_k$ . Moreover, we have

$$\rho \tilde{L}_k = \lambda_k^{-1/2} \rho L_k^* = \lambda_k^{-1/2} \lambda_k^{-1} L_k^* \rho = \lambda_k^{-1} \tilde{L}_k \rho$$

and, in the same way  $\rho \tilde{L}_k^* \rho^{-1} = \lambda_k^{-1} \tilde{L}_k^*$ . Therefore we found a privileged GKSL representation of  $\tilde{\mathcal{L}}$  by means of the operators  $\tilde{H}, \tilde{L}_k$ . (q.e.d.)

## 5 Quantum detailed balance

In this section we characterise the generator of a uniformly continuous QMS satisfying the quantum detailed balance condition.

**Definition 27** A QMS  $\mathcal{T}$  on  $\mathcal{B}(\mathfrak{h})$  satisfies the quantum  $s$ -detailed balance condition ( $s$ -DB) with respect to a normal faithful invariant state  $\rho$ , if its generator  $\mathcal{L}$  and the generator  $\tilde{\mathcal{L}}$  of the  $s$ -dual semigroup  $\tilde{\mathcal{T}}$  satisfy

$$\mathcal{L}(a) - \tilde{\mathcal{L}}(a) = 2i [K, a] \quad (23)$$

with a bounded self-adjoint operator  $K$  on  $\mathfrak{h}$  for all  $a \in \mathcal{B}(\mathfrak{h})$ .

This definition generalises the concept of classical detailed balance discussed in Section 2. Indeed, a classical Markov semigroup  $T$  satisfies the classical detailed balance condition if and only if  $T = \tilde{T}$ , i.e. the generators  $A$  and  $\tilde{A}$  coincide.

**Lemma 28** If  $\mathcal{T}$  satisfies the quantum  $s$ -detailed balance condition then  $\tilde{\mathcal{T}}$  is a QMS and the self-adjoint operator  $K$  in (23) commutes with  $\rho$ .

**Proof.** The identity (23) implies that  $\tilde{\mathcal{L}}$  is conditionally completely positive. Therefore  $\tilde{\mathcal{T}}$  is a QMS. Moreover, recalling that  $\rho$  is an invariant state for both  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  by Proposition 4, for any  $a \in \mathcal{B}(\mathfrak{h})$ , we have then

$$0 = \text{tr} \left( \rho (\mathcal{L}(a) - \tilde{\mathcal{L}}(a)) \right) = 2i \text{tr}(\rho [K, a]) = 2i \text{tr}([\rho, K]a),$$

i.e.  $[K, \rho] = 0$ . This completes the proof. (q.e.d.)

Notice  $[K, \rho] = 0$  and equation (23) imply that the linear operator  $\mathcal{L}' = \mathcal{L} - i[K, \cdot]$  is self-adjoint with respect to the scalar product  $\langle \cdot, \cdot \rangle_0$  on  $\mathcal{B}(\mathfrak{h})$ .

Throughout this section we consider the duality with  $s = 0$ .

**Proposition 29** Given a special GKSL representation of the generator  $\mathcal{L}$  of a QMS  $\mathcal{T}$  by means of operators  $H, L_k$ . Define

$$\mathcal{L}_0(a) = -\frac{1}{2} \sum_k (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k).$$

The QMS  $\mathcal{T}$  satisfies the quantum 0-detailed balance condition if and only if  $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]$  with  $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$  and  $[H, \rho] = 0$ .

**Proof.** Clearly, if  $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]$  with  $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$  and  $[H, \rho] = 0$ , the QMS  $\mathcal{T}$  satisfies the 0-DB. Indeed, if  $\mathcal{L}_0$  is self-adjoint and  $H$  commutes with  $\rho$ , we have  $\tilde{\mathcal{L}} = \mathcal{L}_0 - i[H, \cdot]$ . Therefore  $\mathcal{L}(a) - \tilde{\mathcal{L}}(a) = 2i[H, a]$ .

Conversely, if  $\mathcal{T}$  satisfies the 0-DB condition can find a privileged GKSL of  $\mathcal{L}$  by means of operators  $K, M_k$  by Theorem 24. Note that  $K$  commutes with  $\rho$  because it is the Hamiltonian in a *privileged* GKSL representation. On the other hand, the Hamiltonian  $K$  in a *special* GKSL representation is unique up to a scalar multiple of the identity by Theorem 9, therefore we can take  $H = K$  and we know that: 1)  $H$  commutes with  $\rho$ , 2) the operators  $L_k$  and  $M_k$  define the same map  $\mathcal{L}_0$ .

It follows that  $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]$  and then  $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_0 - i[H, \cdot]$ . Moreover,  $\mathcal{T}$  satisfies the 0-DB condition so that  $\mathcal{L} = \tilde{\mathcal{L}}$ . It follows that  $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ . (q.e.d.)

We can now characterise generators  $\mathcal{L}$  of QMS satisfying the 0-DB condition.

**Theorem 30** *A QMS  $\mathcal{T}$  satisfies the 0-detailed balance condition  $\mathcal{L} - \tilde{\mathcal{L}} = 2i[K, \cdot]$  if and only if there exists a privileged GKSL representation of  $\mathcal{L}$ , by means of operators  $H, L_k$ , such that:*

1.  $H = K + c$  for some  $c \in \mathbb{R}$ ,
2.  $\lambda_k^{-1/2} L_k^* = \sum_j u_{kj} L_j$  for some  $\lambda_k > 0$  and some unitary operator  $(u_{kj})_{kj}$  on  $\mathbf{k}$ .

In particular both  $H$  and  $\sum_k L_k^* L_k$  commute with  $\rho$ .

**Proof.** If  $\mathcal{T}$  satisfies the 0-DB condition its generator  $\mathcal{L}$  and the generator  $\tilde{\mathcal{L}}$  of the dual QMS satisfy  $\mathcal{L}(a) - i[K, a] = \tilde{\mathcal{L}}(a) + i[K, a]$ . Let  $H, L_k$  be the operators in a privileged GKSL representation of  $\mathcal{L}$ . By Theorem 26, the operators  $\tilde{H} = -H - c$  and  $\tilde{L}_k = \lambda_k^{-1/2} L_k^*$  give us a privileged GKSL representation of  $\tilde{\mathcal{L}}$ .

It follows that the operators  $H - K, L_k$  and  $-H + K - c, \lambda_k^{-1/2} L_k^*$  arise in a special GKSL representation of  $\mathcal{L}(\cdot) - i[H, \cdot]$ . Therefore, by Theorem 9,  $H - K = -H + K - c'$  leading us to (1) and there exists a unitary operator  $(u_{kj})_{kj}$  on  $\mathbf{k}$  such that (2) holds.

Conversely if conditions (1) and (2) hold, writing  $\mathcal{L}(a) = \mathcal{L}_0(a) + i[H, a]$ , a straightforward computation shows that  $\text{tr}(\rho \tilde{\mathcal{L}}(a)b) = \text{tr}(\rho a \mathcal{L}(b))$  with  $\tilde{\mathcal{L}}(a) = \mathcal{L}_0(a) - i[H, a]$ . We have then  $\mathcal{L}(a) - \tilde{\mathcal{L}}(a) = 2i[H, a]$  and the 0-DB condition holds with  $K = H$ . (q.e.d.)

**Remark 31** The proof also shows that we can replace “there exists a privileged GKSL ...” by “for every privileged GKSL ...” in Theorem 26.

We conclude this section by showing an example of a QMS  $\mathcal{T}$  whose  $s$ -dual semigroup  $\tilde{\mathcal{T}}$  is still a QMS but does not satisfy the  $s$ -detailed balance condition.

**Example 32** We consider  $\mathbf{h} = \ell^2(\mathbb{Z}_n, \mathbb{C})$ ,  $n \geq 3$ , with the orthonormal basis  $(e_j)_{j=1, \dots, n}$ , and define

$$\mathcal{L}(a) = S^* a S - a,$$

where  $S$  is the unitary shift operator on  $\ell^2(\mathbb{Z}_n)$ ,  $S e_j = e_{j+1}$  (sum modulo  $n$ ).

The QMS  $\mathcal{T}$  generated by  $\mathcal{L}$  admits  $\rho = n^{-1} \mathbf{1}$  as a faithful invariant state because  $\mathcal{L}_*(\mathbf{1}) = S S^* - \mathbf{1} = 0$ . A straightforward computation shows that the dual semigroup  $\tilde{\mathcal{T}}$  is the QMS generated by the linear map  $\tilde{\mathcal{L}}$  defined by  $\tilde{\mathcal{L}}(a) = S a S^* - a$ .

We now check that  $\mathcal{T}$  does not satisfy the 0-detailed balance condition.

Letting  $H = 0$  and  $L_1 = S$  we find a privileged GKSL representation of  $\mathcal{L}$ . Suppose that  $\mathcal{T}$  satisfies the 0-detailed balance condition. Then, by Theorem 26

(1),  $\mathcal{L} = \tilde{\mathcal{L}}$  because  $K$  is a multiple of the identity operator. This identity, however, is not true since

$$\mathcal{L}(|e_2\rangle\langle e_2|) - \tilde{\mathcal{L}}(|e_2\rangle\langle e_2|) = |e_1\rangle\langle e_1| - |e_3\rangle\langle e_3| \neq 0.$$

Note that the condition  $n \geq 3$  is necessary. Indeed, for  $n = 2$ , we can easily check that  $\mathcal{L} = \tilde{\mathcal{L}}$  and the  $s$ -detailed balance condition holds for all  $s \in [0, 1]$ .

## 6 Quantum Markov semigroups on $M_2(\mathbb{C})$

In this section we study in detail the case  $\mathfrak{h} = \mathbb{C}^2$  and  $\mathcal{B}(\mathfrak{h}) = M_2(\mathbb{C})$ . We establish the general form of the generator of a QMS  $\mathcal{T}$  whose 0-dual  $\tilde{\mathcal{T}}$  is a QMS and show that, in this case,  $\mathcal{T}$  satisfies the 0-detailed balance condition.

This can be viewed as a non-commutative counterpart of a well-known fact: any 2-state classical Markov chain satisfies the classical detailed balance condition.

We consider, as usual, the basis  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$  of  $M_2(\mathbb{C})$ , where

$$\sigma_0 = \mathbf{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. Any state on  $M_2(\mathbb{C})$  has the form

$$\frac{1}{2}(\sigma_0 + u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3)$$

for some vector  $(u_1, u_2, u_3)$ , in the unit ball of  $\mathbb{R}^3$ . This state is faithful if the vector  $(u_1, u_2, u_3)$  belongs to the interior of the unit ball, i.e.  $u_1^2 + u_2^2 + u_3^2 < 1$ . After a suitable change of coordinates then we can write a faithful state as

$$\rho = \begin{pmatrix} \nu & 0 \\ 0 & 1 - \nu \end{pmatrix} = \frac{1}{2}(\sigma_0 + (2\nu - 1)\sigma_3)$$

for some  $0 < \nu < 1$ .

We can now characterise special GKSL representations of the generator  $\mathcal{L}$  of a QMS on  $M_2(\mathbb{C})$  in the following way

**Lemma 33** *If  $L_k = \sum_{j=0}^3 z_{kj}\sigma_j$  with  $z_{kj} \in \mathbb{C}$ ,  $k \in \mathcal{J} \subseteq \mathbb{N}$ , then*

$$\mathcal{L}(a) = i[H, a] - \frac{1}{2} \sum_{k \in \mathcal{J}} (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k)$$

*is a special GKSL representation of  $\mathcal{L}$  with respect to  $\rho$  if and only if*

1.  $L_k = -(2\nu - 1)z_{k3}\mathbf{1} + \sum_{j=1}^3 z_{kj}\sigma_j$  for all  $k \in \mathcal{J}$ ,
2.  $\text{card}(\mathcal{J}) \leq 3$  and  $\{z_k : k \in \mathcal{J}\}$  (with  $z_k = (z_{k1}, z_{k2}, z_{k3})$ ) is a set of linearly independent vectors in  $\mathbb{C}^3$ .

**Proof.** A simple calculation shows that  $\text{tr}(\rho L_k) = 2(z_{k0} + (2\nu - 1)z_{k3})$  thus, the condition  $\text{tr}(\rho L_k) = 0$  is equivalent to  $z_{k0} = -(2\nu - 1)z_{k3}$ .

Finally,  $\{\mathbb{1}, L_k : k \in \mathcal{J}\}$  is a set of linearly independent elements in  $M_2(\mathbb{C})$  if and only if the vectors of coefficients w.r.t. the Pauli matrices

$$\{(1, 0, 0, 0), (-(2\nu - 1)z_{k3}, z_{k1}, z_{k2}, z_{k3}) : k \in \mathcal{J}\}$$

are linearly independent in  $\mathbb{C}^4$ ; this is clearly equivalent to have  $\text{card}(\mathcal{J}) \leq 3$  and  $\{z_k : k \in \mathcal{J}\}$  linearly independent on  $\mathbb{C}^3$ ,  $z_k := (z_{k1}, z_{k2}, z_{k3})$ . (q.e.d.)

**Theorem 34** Suppose  $\nu \neq 1/2$  (i.e.  $\rho \neq \mathbb{1}/2$ ) and  $\rho$  invariant for  $\mathcal{T}$ . Then  $\tilde{\mathcal{T}}$  is a QMS if and only if the special Lindblad representation of  $\mathcal{L}$  has the form

$$\begin{aligned} \mathcal{L}(a) = & i[H, a] - \frac{|\eta|^2}{2} (L^2 a - 2LaL + aL^2) \\ & - \frac{|\lambda|^2}{2} (\sigma^- \sigma^+ a - 2\sigma^- a \sigma^+ + a \sigma^- \sigma^+) - \frac{|\mu|^2}{2} (\sigma^+ \sigma^- a - 2\sigma^+ a \sigma^- + a \sigma^+ \sigma^-) \end{aligned} \quad (24)$$

where

$$\begin{aligned} H &= v_0 \sigma_0 + v_3 \sigma_3 = (v_0 + v_3) \sigma^+ \sigma^- + (v_0 - v_3) \sigma^- \sigma^+, \\ L &= -(2\nu - 1) \mathbb{1} + \sigma_3 = (1 - \nu) \sigma^+ - \nu \sigma^-, \end{aligned}$$

$\sigma^+ = \sigma_1 + i\sigma_2$ ,  $\sigma^- = \sigma_1 - i\sigma_2$ ,  $v_0, v_1 \in \mathbb{R}$  and  $\lambda, \mu, \eta \in \mathbb{C}$  satisfy

$$|\lambda|^2 / |\mu|^2 = \nu / (1 - \nu). \quad (26)$$

**Proof.** Consider a special GKSL representation

$$\mathcal{L}(a) = i[H, a] - \frac{1}{2} \sum_{k \in \mathcal{J}} (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k)$$

of  $\mathcal{L}$  with respect to  $\rho$ , where  $\mathcal{J} \subseteq \{1, 2, 3\}$ ,  $H = \sum_{j=0}^3 v_j \sigma_j$  and

$$L_k = -(2\nu - 1) \mathbb{1} + \sum_{j=1}^3 z_{kj} \sigma_j = \begin{pmatrix} 2(1 - \nu)z_{k3} & (z_{k1} - iz_{k2}) \\ (z_{k1} + iz_{k2}) & -2\nu z_{k3} \end{pmatrix},$$

$\{z_k : k \in \mathcal{J}\}$  linearly independent (Lemma 33).

We must find  $v_j$  and  $z_{kj}$  such that:

1.  $[H, \rho] = 0$ ;
2.  $\rho L_k \rho^{-1} = \lambda_k L_k$  and  $\rho L_k^* \rho^{-1} = \lambda_k L_k^*$  for some  $\lambda_k > 0$ ;
3.  $\rho$  is  $\mathcal{T}$ -invariant.

(1) Clearly  $H$  commutes with  $\rho$  if and only if  $v_1 = v_2 = 0$ , i.e.

$$H = v_0 \mathbf{1} + v_3 \sigma_3 = (v_0 + v_3) \sigma^+ \sigma^- + (v_0 - v_3) \sigma^- \sigma^+.$$

(2) Fix  $k \in \mathcal{J}$ . One can easily check that

$$\rho L_k \rho^{-1} = \begin{pmatrix} 2(1-\nu)z_{k3} & \frac{\nu}{(1-\nu)}(z_{k1} - iz_{k2}) \\ \frac{1-\nu}{\nu}(z_{k1} + iz_{k2}) & -2\nu z_{k3} \end{pmatrix},$$

and, since  $\nu \neq 1/2$ , the identity  $\rho L_k \rho^{-1} = \lambda_k L_k$  holds if and only if either

$$\begin{cases} \lambda_k = 1 \\ z_{k1} - iz_{k2} = 0 \\ z_{k1} + iz_{k2} = 0, \end{cases} \quad \text{or} \quad \begin{cases} z_{k3} = 0 \\ \left(\frac{\nu}{1-\nu} - \lambda_k\right)(z_{k1} - iz_{k2}) = 0 \\ \left(\frac{1-\nu}{\nu} - \lambda_k\right)(z_{k1} + iz_{k2}) = 0, \end{cases} \quad (27)$$

In the first case, we get  $L_k = z_{k3}(-(2\nu-1)\mathbf{1} + \sigma_3) = z_{k3}((1-\nu)\sigma^+ - \nu\sigma^-)$ ; since  $\{L_k : k \in \mathcal{J}\}$  is a set of linearly independent elements in  $M_2(\mathbb{C})$ , this means that there exists a unique  $k_0 \in \mathcal{J}$  such that  $\lambda_{k_0} = 1$ . We can suppose  $k_0 = 3$ .

Therefore, for  $k = 1, 2$ , conditions (27) are equivalent to

$$\begin{cases} z_{k3} = 0 \\ \frac{\nu}{1-\nu} = \lambda_k \\ z_{k1} + iz_{k2} = 0 \end{cases} \quad \text{or} \quad \begin{cases} z_{k3} = 0 \\ z_{k1} - iz_{k2} = 0 \\ \frac{1-\nu}{\nu} = \lambda_k, \end{cases}$$

that is

$$L_k = \begin{pmatrix} 0 & -iz_{k2} \\ 0 & 0 \end{pmatrix} = -iz_{k2} \sigma^+ \quad \text{and} \quad \lambda_k = \frac{\nu}{1-\nu},$$

or

$$L_k = \begin{pmatrix} 0 & 0 \\ iz_{k2} & 0 \end{pmatrix} = iz_{k2} \sigma^- \quad \text{and} \quad \lambda_k = \frac{1-\nu}{\nu},$$

so that we have  $L_1 = -iz_{12} \sigma^+ = \lambda \sigma^+$  and  $L_2 = iz_{22} \sigma^- = \mu \sigma^-$ , with  $\lambda_1 = \nu/(1-\nu)$  and  $\lambda_2 = \lambda_1^{-1}$ .

Moreover, with this choice of  $L_1, L_2$  and  $L_3$ , the equalities  $\rho L_k^* \rho^{-1} = \lambda_k^{-1} L_k^*$  are automatically satisfied.

(3) Since  $H, L_3$  and  $\rho$  commute,  $\rho$  is  $\mathcal{T}$ -invariant if and only if

$$\begin{aligned} 0 &= \frac{1}{2} \sum_{k=1}^2 (L_k^* L_k \rho - 2L_k \rho L_k^* + \rho L_k^* L_k) \\ &= \frac{1}{2} \sum_{k=1}^2 (L_k^* L_k \rho - 2L_k (\rho L_k^* \rho^{-1}) \rho + (\rho L_k^* \rho^{-1}) (\rho L_k \rho^{-1}) \rho) \\ &= \sum_{k=1}^2 (L_k^* L_k \rho - \lambda_k^{-1} L_k L_k^*), \end{aligned}$$

that is

$$\frac{|\lambda|^2}{|\mu|^2} = \frac{|z_{12}|^2}{|z_{22}|^2} = \frac{\nu}{1-\nu}.$$

This concludes the proof.

(q.e.d.)

**Theorem 35** *Suppose  $\nu \neq 1/2$ .*

*If  $\tilde{\mathcal{T}}$  is a QMS, then  $\mathcal{T}$  satisfies detailed balance.*

**Proof.** By Theorem 34 there exists a privileged GKSL representation of  $\mathcal{L}$  with

$$\begin{cases} L_1 = \eta L \\ L_2 = \lambda \sigma^+ \\ L_3 = \mu \sigma^- \end{cases} \quad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \frac{\nu}{1-\nu} \\ \lambda_3 = \lambda_2^{-1} \end{cases} \quad \text{and} \quad \frac{|\lambda|^2}{|\mu|^2} = \frac{\nu}{1-\nu}. \quad (28)$$

Therefore,

$$\begin{pmatrix} \sqrt{\lambda_1^{-1}} L_1^* \\ \sqrt{\lambda_2^{-1}} L_2^* \\ \sqrt{\lambda_3^{-1}} L_3^* \end{pmatrix} = \begin{pmatrix} \frac{\bar{\eta} L}{\sqrt{\frac{1-\nu}{\nu} \bar{\lambda}} \sigma^-} \\ \sqrt{\frac{\nu}{1-\nu}} \bar{\mu} \sigma^+ \end{pmatrix} = \begin{pmatrix} \eta/\bar{\eta} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1-\nu}{\nu} \frac{\bar{\lambda}}{\mu}} \\ 0 & \sqrt{\frac{1-\nu}{\nu} \frac{\bar{\mu}}{\lambda}} & \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \quad (29)$$

and

$$\begin{pmatrix} \eta/\bar{\eta} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1-\nu}{\nu} \frac{\bar{\lambda}}{\mu}} \\ 0 & \sqrt{\frac{1-\nu}{\nu} \frac{\bar{\mu}}{\lambda}} & \end{pmatrix}$$

is unitary thanks to (28).

(q.e.d.)

## 7 The symmetric dual semigroup and detailed balance condition

We now study the  $s$ -dual semigroup and the quantum  $s$ -detailed balance condition for  $s = 1/2$ . In this case we call  $\mathcal{T}'$  the *symmetric dual semigroup* of  $\mathcal{T}$  and call *symmetric detailed balance* condition the  $1/2$ -detailed balance condition.

By Proposition 4, the symmetric dual semigroup of  $\mathcal{T}$  is defined by

$$\rho^{1/2} \mathcal{T}'_t(a) \rho^{1/2} = \mathcal{T}_{*t}(\rho^{1/2} a \rho^{1/2}), \quad (30)$$

so that

$$\mathcal{T}'_t(a) \supseteq \rho^{-1/2} \mathcal{T}_{*t}(\rho^{1/2} a \rho^{1/2}) \rho^{-1/2}$$

for all  $a \in \mathcal{B}(\mathfrak{h})$ . The name symmetric is then justified by the left-right symmetry of multiplication by  $\rho^{1/2}$  and  $\rho^{-1/2}$ . Equation (30) ensures that any map  $\mathcal{T}'_t$  is completely positive, contrary to the case  $s = 0$  (Example 25). Therefore the symmetric dual semigroup  $\mathcal{T}'$  is always a QMS with generator given by (Proposition 6)

$$\rho^{1/2} \mathcal{L}'(a) \rho^{1/2} = \mathcal{L}_*(\rho^{1/2} a \rho^{1/2}).$$

The relationship between dual semigroups  $\tilde{\mathcal{T}}$  and  $\mathcal{T}'$  is described by the following



**Theorem 36** *The 0-dual  $\tilde{\mathcal{T}}$  and the symmetric dual  $\mathcal{T}'$  of a QMS  $\mathcal{T}$  coincide if and only if each map  $\mathcal{T}_t$  commutes with  $\sigma_{-i}$ .*

**Proof.** If  $\tilde{\mathcal{T}} = \mathcal{T}'$ , then  $\tilde{\mathcal{T}}$  is a QMS; hence, by Theorem 8,  $\mathcal{T}$  commutes with the modular automorphism  $\sigma_{-i}$ .

On the other hand, we showed in the proof of Theorem 8 that the commutation between  $\mathcal{T}$  and  $\sigma_{-i}$  implies  $\tilde{\mathcal{T}}_t(a) = \rho^{-1/2}\mathcal{T}_{*t}(\rho^{1/2}a\rho^{1/2})\rho^{-1/2}$  for all  $a \in \mathcal{B}(\mathfrak{h})$ ,  $t \geq 0$ , and then  $\tilde{\mathcal{T}} = \mathcal{T}'$ . (q.e.d.)

We now establish the relationship between the generator  $\mathcal{L}$  of a QMS and the generator  $\mathcal{L}'$  of the symmetric dual semigroup.

**Theorem 37** *For all special GKSL representation  $\mathcal{L}(a) = G^*a + \sum_k L_k^*aL_k + aG$  of  $\mathcal{L}$  there exists a special GKSL representation of  $\mathcal{L}'$  by means of operators  $G', L'_k$  such that:*

1.  $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$  for some  $c \in \mathbb{R}$ ,
2.  $L'_k\rho^{1/2} = \rho^{1/2}L_k^*$

**Proof.** Since  $\mathcal{T}'$  is a uniformly continuous QMS, its generator  $\mathcal{L}'$  admits a special GKSL representation,  $\mathcal{L}'(a) = G'^*a + \sum_k L'_k{}^*aL'_k + aG'$ . Moreover, by Proposition 15 we have  $G'\rho^{1/2} = \rho^{1/2}G^* + ic$ ,  $c \in \mathbb{R}$ , and so the relation  $\rho^{1/2}\mathcal{L}'(a)\rho^{1/2} = \mathcal{L}_*(\rho^{1/2}a\rho^{1/2})$  implies

$$\sum_k \rho^{1/2}L'_k{}^*aL'_k\rho^{1/2} = \sum_k L_k\rho^{1/2}a\rho^{1/2}L_k^*. \quad (31)$$

Define

$$X(x \otimes \mathbb{1}_{\mathfrak{k}'})L'\rho^{1/2}u = (x \otimes \mathbb{1}_{\mathfrak{k}})(\rho^{1/2} \otimes \mathbb{1}_{\mathfrak{k}})L^*u$$

for all  $x \in \mathcal{B}(\mathfrak{h})$  and  $u \in \mathfrak{h}$ , where  $L : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{k}$ ,  $Lu = \sum_k L_k u \otimes f_k$ ,  $L' : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{k}'$ ,  $L'u = \sum_k L'_k u \otimes f'_k$ ,  $(f_k)_k$  and  $(f'_k)_k$  orthonormal basis of  $\mathfrak{k}$  and  $\mathfrak{k}'$  respectively. Thus, by (31),

$$\begin{aligned} \langle X(x \otimes \mathbb{1}_{\mathfrak{k}'})L'\rho^{1/2}u, X(y \otimes \mathbb{1}_{\mathfrak{k}'})L'\rho^{1/2}v \rangle &= \sum_k \langle u, \rho^{1/2}L'_k{}^*x^*yL'_k\rho^{1/2}v \rangle \\ &= \langle (x \otimes \mathbb{1}_{\mathfrak{k}'})L'\rho^{1/2}u, (y \otimes \mathbb{1}_{\mathfrak{k}'})L'\rho^{1/2}v \rangle \end{aligned}$$

for all  $x, y \in \mathcal{B}(\mathfrak{h})$  and  $u, v \in \mathfrak{h}$ , i.e.  $X$  preserves the scalar product. Therefore, since the set  $\{(x \otimes \mathbb{1}_{\mathfrak{k}'})L'\rho^{1/2}u \mid x \in \mathcal{B}(\mathfrak{h}), u \in \mathfrak{h}\}$  is total in  $\mathfrak{h} \otimes \mathfrak{k}'$  (for  $\rho^{1/2}(\mathfrak{h})$  is dense in  $\mathfrak{h}$  and Theorem 13 holds), we can extend  $X$  to an unitary operator from  $\mathfrak{h} \otimes \mathfrak{k}'$  to  $\mathfrak{h} \otimes \mathfrak{k}$ . As a consequence we have  $X^*X = \mathbb{1}_{\mathfrak{h} \otimes \mathfrak{k}'}$ .

Moreover, since  $X(y \otimes \mathbb{1}_{\mathfrak{k}'}) = (y \otimes \mathbb{1}_{\mathfrak{k}'})X$  for all  $y \in \mathcal{B}(\mathfrak{h})$ , we can conclude that  $X = \mathbb{1}_{\mathfrak{h}} \otimes Y$  for some unitary map  $Y : \mathfrak{k}' \rightarrow \mathfrak{k}$ .

The definition of  $X$  implies then

$$(\rho^{1/2} \otimes \mathbb{1}_{\mathfrak{k}})L^* = XL'\rho^{1/2} = (\mathbb{1}_{\mathfrak{h}} \otimes Y)L'\rho^{1/2}.$$

This means that, by substituting  $L'$  by  $(\mathbb{1}_h \otimes Y)L'$ , or more precisely  $L'_k$  by  $\sum_l u_{kl} L'_l$  for all  $k$ , we have

$$\rho^{1/2} L_k^* = L'_k \rho^{1/2}.$$

Since  $\text{tr}(\rho L'_k) = \text{tr}(\rho L_k^*) = 0$  and, from  $\mathcal{L}'(\mathbb{1}) = 0$  (Proposition 4),  $G'^* + G' = \sum_k L_k^* L'_k$ , the properties of a special GKSL representation follow. (q.e.d.)

Contrary to what happens in the case  $s = 0$ , the operators  $G, G'$  may not commute with  $\rho$ , as the following example shows.

**Example 38** Fix a faithful state  $\rho = (\mathbb{1} + (2\nu - 1)\sigma_3)/2$  on  $M_2(\mathbb{C})$  with  $\nu \in ]0, 1[$ ,  $\nu \neq 1/2$  and consider the semigroup on  $M_2(\mathbb{C})$  generated by

$$\mathcal{L}(a) = i[H, a] - \frac{1}{2}(L^*La - 2L^*aL + aL^*L)$$

with  $H = \Omega\sigma_1$ ,  $L = (1 - 2\nu)\mathbb{1} + ir\sigma_1 + s\sigma_2 + \sigma_3$  and  $\Omega, r, s \in \mathbb{R}$ ,  $\Omega \neq 0$ . Clearly  $\mathcal{L}$  is represented in a special GKSL form with respect to the faithful state  $\rho$  and  $H$  does not commute with  $\rho$ .

We now show that  $\rho$  is an invariant state for the QMS generated by  $\mathcal{L}$  for a special choice of the constants  $\Omega, r, s$  and so we find the desired example.

A long but straightforward computation shows that, if we choose  $r, s$  satisfying

$$2\nu = (r - s)^2 / (r^2 + s^2), \quad (32)$$

then we find  $\mathcal{L}_*(\rho) = ((-4\nu^2 + 4\nu + 1)r + (2\nu - 1)(s - \Omega))\sigma_2$ . It is now a simple exercise to show that for all fixed  $\nu$  and  $\Omega \neq 0$  there exist  $r, s$  satisfying (32) and

$$s = \Omega + r(4\nu^2 - 4\nu - 1) / (2\nu - 1). \quad (33)$$

A little computation yields

$$s = \frac{\pm\Omega}{2\sqrt{\nu(1-\nu)}}, \quad r = \frac{\pm\Omega(1-2\nu)}{2\sqrt{\nu(1-\nu)}(1 \pm 2\sqrt{\nu(1-\nu)})} \quad (34)$$

( $\pm$  are all  $+$  or all  $-$ ). With this choice of  $r$  and  $s$  the state  $\rho$  is invariant.

The 0-detailed balance is stronger than the symmetric detailed balance (see also [5] Th. 6.6 p.296).

**Proposition 39** *If  $\mathcal{T}$  satisfies the 0-detailed balance, then it also fulfills the symmetric detailed balance. Moreover, these conditions are equivalent if and only if the 0-dual  $\tilde{\mathcal{T}}$  is a QMS.*

**Proof.** Suppose that  $\tilde{\mathcal{T}}$  is a QMS. As we showed in the proof of Theorem 8,  $\tilde{\mathcal{T}}_t(a) = \rho^{-1/2} \mathcal{T}_{*t}(\rho^{1/2} a \rho^{1/2}) \rho^{-1/2}$ . Then  $\tilde{\mathcal{T}} = \mathcal{T}'$  by (30), as a consequence  $\tilde{\mathcal{L}} = \mathcal{L}'$ , i.e.  $\mathcal{L} - \tilde{\mathcal{L}} = \mathcal{L} - \mathcal{L}'$ , and both detailed balance conditions are equivalent.

On the other hand, if  $\mathcal{T}$  satisfies the 0-detailed balance, then  $\tilde{\mathcal{T}}$  is a QMS. Therefore  $\tilde{\mathcal{T}} = \mathcal{T}'$  and  $\mathcal{T}$  also fulfills the symmetric detailed balance condition. (q.e.d.)

We end this section by finding the relationships between the operators  $H, L_k$  in a special GKSL representation of the generator of a QMS satisfying symmetric detailed balance.

**Theorem 40** *A QMS  $\mathcal{T}$  satisfies the symmetric detailed balance condition  $\mathcal{L} - \mathcal{L}' = 2i[K, \cdot]$  if and only if there exists a special GKSL representation of the generator  $\mathcal{L}$  by means of operators  $H, L_k$  such that, letting  $2G = -\sum_k L_k^* L_k - 2iH$ , we have:*

1.  $G\rho^{1/2} = \rho^{1/2}G^* - (2iK + ic)\rho^{1/2}$  for some  $c \in \mathbb{R}$ ,
2.  $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$ , for all  $k$ , for some unitary matrix  $(u_{k\ell})_{k\ell}$ .

**Proof.** Choose a special GKSL representation of  $\mathcal{L}$  by means of operators  $H, L_k$ . Theorem 37 allows us to write the dual  $\mathcal{L}'$  in a special GKSL representation by means of operators  $H', L'_k$  with  $H' = (G'^* - G')/(2i)$ ,

$$G'\rho^{1/2} = \rho^{1/2}G^*, \quad L'_k\rho^{1/2} = \rho^{1/2}L_k^*. \quad (35)$$

Suppose first that  $\mathcal{T}$  satisfies the symmetric detailed balance condition. Then  $\mathcal{L} - i[K, \cdot] = \mathcal{L}' + i[K, \cdot]$  and  $K$  commutes with  $\rho$  by Lemma 28. Comparing the special GKSL representations of  $\mathcal{L} - i[K, \cdot]$  and  $\mathcal{L}' + i[K, \cdot]$ , by Theorem 9 and Remark 12 we find

$$G + iK = G' - iK + ic, \quad L'_k = \sum_j u_{kj}L_j,$$

for some unitary matrix  $(u_{kj})_{kj}$  and some  $c \in \mathbb{R}$ . This, together with (35) implies that conditions (1) and (2) hold.

Conversely, notice that the dual  $\mathcal{L}'$  admits the special GKSL representation

$$\mathcal{L}'(a) = G'^*a + \sum_k L_k'^* a L_k' + aG'.$$

Therefore, if conditions (1) and (2) are satisfied, by (35), we have

$$G'\rho^{1/2} = \rho^{1/2}G^* = (G + 2iK)\rho^{1/2},$$

so that  $G' = G + 2iK$  and then

$$\begin{aligned} G'^*a + aG' &= (G^* - 2iK)a + a(G + 2iK) = G^*a + aG - 2i[K, a] \\ \sum_k L_k'^* a L_k' &= \sum_{k,j,m} \bar{u}_{kj}L_j^* a u_{km}L_m = \sum_{j,m} \left( \sum_k \bar{u}_{kj}u_{km} \right) L_j^* a L_m = \sum_k L_k^* a L_k. \end{aligned}$$

It follows that  $\mathcal{L}'(a) = \mathcal{L}(a) - 2i[K, a]$  and the symmetric detailed balance condition holds. (q.e.d.)

The Hamiltonian  $H$  in a special GKSL representation of the generator of a QMS satisfying the *symmetric* detailed balance condition does not need to commute with the invariant state  $\rho$  (as in the case of 0-detailed balance) as shows the following

**Example 41** Let  $\mathcal{L}$  be the generator described in Example 38 and let  $\mathcal{L}'$  be its symmetric dual. The linear map  $\mathcal{K} = (\mathcal{L} + \mathcal{L}')/2$  is clearly the generator of a QMS. Moreover,  $\rho$  is an invariant state for  $\mathcal{K}$  because it is an invariant state for  $\mathcal{L}$  and  $\mathcal{L}'$  by Proposition 4.  $\mathcal{K}$  satisfies the symmetric detailed balance condition by its definition.

The special GKSL representation of  $\mathcal{L}$  by means of operators  $H, L$  as in Example 38 yields a special GKSL representation of  $\mathcal{L}'$  choosing  $L' = \rho^{1/2}L^*\rho^{-1/2}$  and  $H' = (G'^* - G')/(2i)$  with  $G' = \rho^{1/2}G^*\rho^{-1/2}$  and  $2G = -L^*L - iH$ . Putting

$$\begin{aligned} M_1 &= L/\sqrt{2}, & M_2 &= \rho^{1/2}L^*\rho^{-1/2}/\sqrt{2}, \\ F &= (G + G')/2, & F_0 &= (F + F^*)/2, \quad K = (F^* - F)/(2i) \end{aligned}$$

we have  $2F_0 = M_1^*M_1 + M_2^*M_2$  and a special GKSL representation of  $\mathcal{K}$  by means of operators  $K, M_j$ .

We now check that  $K$  does not commute with  $\rho$ . To this end it suffices that  $K$  is not linearly independent of  $\sigma_j$  for  $j = 1$  or  $j = 2$ , namely  $\text{tr}(\sigma_j K) \neq 0$ . But

$$2\text{tr}(\sigma_j K) = 2\Im\text{tr}(\sigma_j F) = \Im\text{tr}(\sigma_j(G + G')) = \Im\text{tr}(\sigma_j G) + \Im\text{tr}(\rho^{-1/2}\sigma_j\rho^{1/2}G^*),$$

where, defining  $r$  and  $s$  as in (34) with  $-$  signs and computing  $(2\nu - 1)s + r = (2\nu - 1)\Omega/(1 - 2\sqrt{\nu(1 - \nu)})$  we have

$$G = -\frac{(1 - 2\nu)^2 + 1 + s^2 + r^2}{2}\mathbf{1} - i\Omega\sigma_1 + \frac{(2\nu - 1)\Omega}{1 - 2\sqrt{\nu(1 - \nu)}}\sigma_2 + ((2\nu - 1) - rs)\sigma_3.$$

Another straightforward computation yields

$$\rho^{1/2} = \frac{\kappa}{2}\left(\mathbf{1} + \frac{2\nu - 1}{\kappa^2}\sigma_3\right), \quad \rho^{-1/2} = \frac{2}{\kappa}\frac{1}{1 - \kappa^{-4}(2\nu - 1)^2}\left(\mathbf{1} - \frac{2\nu - 1}{\kappa^2}\sigma_3\right)$$

where  $\kappa := \sqrt{1 + 2\sqrt{\nu(1 - \nu)}} = \sqrt{\nu} + \sqrt{1 - \nu}$  and

$$\rho^{-1/2}\sigma_1\rho^{1/2} = \frac{1}{2\sqrt{\nu(1 - \nu)}}(\sigma_1 - i(2\nu - 1)\sigma_2).$$

It follows that  $2\text{tr}(\sigma_1 K) = -2\Omega$ . Therefore we find  $\text{tr}(\sigma_1 K) \neq 0$  for all  $\nu \in ]0, 1[$  with  $\nu \neq 1/2$  and  $\Omega \neq 0$ .

## 8 Case $s \in (0, 1/2) \cup (1/2, 1)$

We conclude the discussion on the  $s$ -dual semigroup by considering  $s \in (0, 1/2) \cup (1/2, 1)$ . In this framework, we show that  $\tilde{\mathcal{T}}^{(s)}$  is a QMS if and only if the 0-dual semigroup is a QMS and, in this case, they coincide. Therefore, it is enough to study the case  $s = 0$ .

**Proposition 42** *The following facts are equivalent:*

1.  $\tilde{\mathcal{T}}^{(s)}$  is a QMS;
2.  $\tilde{\mathcal{T}}^{(0)}$  is a QMS.

Moreover, if the above conditions hold, then  $\tilde{\mathcal{T}}^{(s)} = \tilde{\mathcal{T}}^{(0)}$ .

**Proof.**  $1 \Rightarrow 2$ . Since  $\tilde{\mathcal{T}}^{(s)t}$  and  $\mathcal{T}_{*t}$  are  $*$ -maps, by the second formula (9) and the same formula taking the adjoint we have

$$\rho^s \mathcal{T}_t(a) \rho^{1-s} = \tilde{\mathcal{T}}_t^{(s)}(\rho^s a \rho^{1-s}) \quad \text{and} \quad \rho^{1-s} \mathcal{T}_t(a) \rho^s = \tilde{\mathcal{T}}_t^{(s)}(\rho^{1-s} a \rho^s).$$

Therefore, given  $a \in \mathcal{A}$ , we get

$$\rho^s \mathcal{T}_t(a) \rho^{1-s} = \tilde{\mathcal{T}}_t^{(s)}(\rho^{1-s}(\rho^{2s-1} a \rho^{1-2s}) \rho^s) = \rho^{1-s} \mathcal{T}_t(\rho^{2s-1} a \rho^{1-2s}) \rho^s$$

and then

$$\mathcal{T}_t(a) = \rho^{1-2s} \mathcal{T}_t(\rho^{2s-1} a \rho^{1-2s}) \rho^{2s-1}$$

i.e. any  $\mathcal{T}_t$  commutes with  $\sigma_{-i(2s-1)}$ .

This means that the contraction semigroup  $(\hat{\mathcal{T}}_t)$  defined on  $L^2(\mathfrak{h})$  by

$$\hat{\mathcal{T}}_t(a \rho^{1/2}) = \mathcal{T}_t(a) \rho^{1/2}$$

commutes with  $\Delta^{1-2s}$  and then, by spectral calculus, it also commutes with  $\rho$ ; it follows that  $\mathcal{T}_t$  commutes with the modular automorphism  $\sigma_{-i}$  and so  $\tilde{\mathcal{T}}^{(0)}$  is a QMS by Theorem 8.

$2 \Rightarrow 1$ . If  $\tilde{\mathcal{T}}^{(0)}$  is a QMS, by Theorem 24 there exists a privileged GKSL representation of  $\mathcal{L}$  by means of operators  $H$  and  $L_k$  such that

$$\Delta(L_k \rho^{1/2}) = \rho L_k \rho^{-1/2} = \lambda_k L_k \rho^{1/2} \quad \text{and} \quad \Delta(L_k^* \rho^{1/2}) = \rho L_k^* \rho^{-1/2} = \lambda_k^{-1} L_k^* \rho^{1/2}.$$

It follows by spectral calculus that

$$\begin{aligned} \rho^\alpha L_k \rho^{-\alpha} \rho^{1/2} &= \Delta^\alpha(L_k \rho^{1/2}) = \lambda_k^\alpha L_k \rho^{1/2} \\ \rho^\alpha L_k^* \rho^{-\alpha} \rho^{1/2} &= \Delta^\alpha(L_k^* \rho^{1/2}) = \lambda_k^{-\alpha} L_k^* \rho^{1/2} \end{aligned}$$

for all  $\alpha \neq 0$ .

Therefore, since  $H$  and  $\rho$  commute, we have

$$\begin{aligned}\tilde{\mathcal{L}}^{(s)}(a) &= \rho^{s-1} \mathcal{L}_*(\rho^{1-s} a \rho^s) \rho^{-s} = -i \rho^{s-1} [H, \rho^{1-s} a \rho^s] \rho^{-s} \\ &- \frac{1}{2} \sum_k (\rho^{s-1} L_k^* L_k \rho^{1-s} a - 2 \rho^{s-1} L_k \rho^{1-s} a \rho^s L_k^* \rho^{-s} + a \rho^s L_k^* L_k \rho^{-s}) \\ &= -i[H, a] - \frac{1}{2} \sum_k (L_k^* L_k a - 2 \lambda_k^{-1} L_k a L_k^* + a L_k^* L_k)\end{aligned}$$

for all  $a \in \mathcal{A}$ ; since  $-i[H, a] - \frac{1}{2} \sum_k (L_k^* L_k a - 2 \lambda_k^{-1} L_k a L_k^* + a L_k^* L_k) = \tilde{\mathcal{L}}^{(0)}(a)$  by Theorem 26 and  $\mathcal{A}$  is  $\sigma$ -weakly dense in  $\mathcal{B}(\mathfrak{h})$ , the above equality means  $\tilde{\mathcal{L}}^{(s)} = \tilde{\mathcal{L}}^{(0)}$ , so that  $\tilde{T}^{(s)}$  is a QMS and it coincides with  $\tilde{T}^{(0)}$ . (q.e.d.)

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